

申龍秀 教授指導
碩士學位 請求論文

THE EXAMPLE OF A NON LEVEL
O-SEQUENCE OF CODIMENSION 3
ENDING 6, 4, 2

2006

誠信女子大學校 教育大學院

教育學科 數學教育專攻

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Abstract

논문개요

참고논문 7번에서 A. V. Geramita, T. Harima, J. C. Migliore 그리고 Y. S. Shin은 여차원 3인 O-수열이

$$1, 3, \dots, 6, 3, 2$$

또는

$$1, 3, \dots, 5, 3, 2$$

형태이면 Level O-수열이 되지 않는다는 것을 증명하였다.

본 논문에서는 Inverse System (또는 Macaulay Duality)를 이용하여 여차원 3인 O-수열

$$1, 3, 4, 5, \dots, 6, 4, 2$$

가 Level이 되지 않는다는 것을 증명하였다.

1. INTRODUCTION

Let $R = k[x_0, x_1, \dots, x_n] = \bigoplus_{i \geq 0} R_i$, k an algebraically closed field of characteristic 0, and let I be a homogeneous ideal of R , $A = R/I$. The **Hilbert function** of A , $\mathbf{H}_A : \mathbb{N} \rightarrow \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R . The **h -vector** of A is $h(A) = (h_0, h_1, \dots, h_s)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and s is the last index such that $\dim_k A_k \neq 0$. We call s the **socle degree** of A . Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the **codimension** of A .

Let $R = k[x_0, \dots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d . Let

$$0 \rightarrow \mathcal{F}_{n-(d-1)} \rightarrow \dots \rightarrow \mathcal{F}_1 \rightarrow I \rightarrow 0$$

be a minimal free resolution of I . A is a **level algebra** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$.

The sequence $\{h_i\}_{i \geq 0}$ (with $h_0 = 1$ and $h_1 \leq n$) is called an **\mathcal{O} -sequence** if there is a homogeneous ideal $I \subset R$ such that if $A = R/I$ then $\mathbf{H}_A(i) = h_i$. In particular, the h -vector $h = (1, n, h_2, \dots, h_s)$ is called an **\mathcal{O} -sequence** if there is an Artinian quotient A of R whose h -vector is h . The h -vector $h = (1, n, h_2, \dots, h_s)$ is called a **level \mathcal{O} -sequence** if there is a level Artinian algebra having h -vector h (see

[1]–[5], [7]–[9], [11], [13], [14]). Moreover, we say that the sequence is a **Gorenstein sequence** if it is a level sequence with $h_s = 1$ (see [2], [4], [9]). When $h_s \neq 0$ we say that $s + 1$ is the **length** of the sequence.

For graded Artinian level algebras, it has been recently studied (see [2], [5], [7], [8], [11], [13], [14]).

Let A be a ring and M be an A -module. An element a of A is said **M -regular** if it is not a zero-divisor on M , i.e., if $M \xrightarrow{a} M$ is injective. The set of the M -regular elements is a multiplicative subset of A .

We say a_1, a_2, \dots, a_r is an **M -regular sequence** (or simply M -sequence) if the following conditions are satisfied:

- (1) $\forall 1 \leq i \leq r$, a_i is not a zero-divisor on $M/(a_1, a_2, \dots, a_{i-1})M$,
- (2) $M \neq aM$.

When all a_i belong to an ideal I we say a_1, a_2, \dots, a_r is an **M -regular sequence** in I .

Let I be a lex-segment ideal of $R = k[x, y, z]$ with Hilbert function $\mathbf{H} = (1, 3, 4, 5, 6, 4, 2)$. Then the minimal free resolution of R/I is

$$0 \rightarrow R^2(-7) \oplus R^2(-8) \oplus R^2(-9) \rightarrow R(-3) \oplus R^5(-6) \oplus R^4(-7) \oplus R^4(-8) \rightarrow R^2(-2) \oplus R^3(-5) \oplus R^2(-6) \oplus R^2(-7) \rightarrow R_i \rightarrow R/I \rightarrow 0.$$

As we see from the minimal free resolution of R/I above, we cannot say if the Hilbert function \mathbf{H} is level or not.

A total order on the monomials of each degree is said to be a **term order** if

- (1) $x_1 > \cdots > x_n$, and
- (2) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m , m_1 and m_2 .

The **lexicographic order** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

- (1) $\sum i_t > \sum j_t$ or
- (2) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.

Let S be a subset of all monomials in R . S is a **lex-segment** if a monomial m of degree d is in S , then every monomial m' of degree d in R_d such that $m' > m$ is in S . Let $I = \bigoplus_{t \geq 0} I_t$ be a graded ideal of R . We say that I is a **lex-segment ideal** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

The goal of this thesis is to prove that the h -vector $(1, 3, 4, 5, \dots, 6, 4, 2)$ is not level (see Theorem 7) in general using the theory of **inverse systems** or **Macaulay duality** (see [6] for more details).

2. INVERSE SYSTEMS

We now recall a very interesting method for constructing Artinian level algebras. This method is based on the idea of **Macaulay's Inverse Systems**. We will only give a quick review of the method and refer the reader [6] for more details.

Let $R = k[x_1, \dots, x_n]$ and $S = k[y_1, \dots, y_n]$. We can consider S as a graded R -module by: if $F \in S_j$ then $x_i \circ F = (\frac{\partial}{\partial y_i})F$. We extend this action in the obvious way and note that the action *lowers* degree on S and hence S is not a finitely generated R -module.

There is an order reversing function from the ideals of R to the R -submodules of S defined by:

$$\varphi_1 : \{\text{ideals of } R\} \rightarrow \{R\text{-submodules of } S\}$$

where

$$\varphi_1(I) = \{F \in S \mid G \circ F = 0 \text{ for all } G \in I\}$$

This is a 1-1 correspondence whose inverse (φ_2) is given by $\varphi_2(M) = \text{ann}_R(M) = \{r \in R \mid r \cdot x = 0, \forall x \in M\}$. In fact, we denote $\varphi_1(I)$ by I^{-1} , which is called the **inverse system** to I .

It is very easy to construct I^{-1} (and this is at the heart of the proof of the 1-1 correspondence). One first observes that the pairing

$$R_j \times S_j \longrightarrow S_0 \simeq k$$

is a perfect pairing and so S_j can be identified with R_j^* (the dual vector space to R_j). If V is a subspace of R_j we write V^\perp for the annihilator of V in this pairing. Then, if $I \subset R$ is an ideal and I_j its j^{th} graded piece, then Macaulay observed that:

$$(I^{-1})_j = I_j^\perp.$$

It follows immediately that

$$\dim_k(I^{-1})_j = \dim_k R_j - \dim_k I_j = \mathbf{H}(R/I, j).$$

It is a simple consequence of this last observation that I^{-1} is a finitely generated R -submodule of S if and only if R/I is Artinian.

Remark 2.1. There is another way to define Inverse Systems which considers S as an R -module in a different way. In this other method, we consider the *contraction* operations, D_{x_i} where, if F is a monomial in S_j then

$$D_{x_i}(F) = \begin{cases} 0, & \text{if } y_i \text{ does not divide } F, \\ F/y_i & \text{if } y_i \text{ divides } F. \end{cases}$$

We extend this action to all of S in the obvious way and recall that when the characteristic of k is 0, this action is equivalent to the one described above. The contraction operation has the advantage that it doesn't end up increasing the sizes of coefficients (see [6] for more details).

The really interesting connection between inverse systems and what we've been considering is the following theorem of Macaulay. We continue with notations as above.

Theorem 2.2 (Macaulay). *Let I be an Artinian ideal of R and I^{-1} its inverse system. Then I^{-1} has exactly ν_j minimal generators of degree j if and only if the socle of R/I in degree j has dimension exactly ν_j .*

Remark 2.3. 1) This gives us a new interpretation of the socle vector of an Artinian algebra of the form $A = k[x_1, \dots, x_n]/I$. The entries of the socle vector tell us the number of generators of the inverse system of I in each degree.

2) Since we are interested in level algebras (Artinian, say, with socle degree s , type c and embedding dimension n) then this theorem tells us how to make **all** of them. We look at every subspace of $S_s = k[y_1, \dots, y_n]_s$ of dimension c and form the R -submodule of S generated by that subspace. The result is a level algebra of the type we are looking for and every level algebra of socle degree s , type c and embedding dimension n arises in this way.

Example 2.4. suppose we would like to construct a level algebra with socle degree 4, embedding dimension 3 and type 2. Macaulay's Theorem says we have to look at a two dimensional vector space of S_4 , where $S = k[y_1, y_2, y_3]$ and take the inverse system it generates.

For example, consider the vector space of S_4 generated by $F_1 = y_1^4$ and $F_2 = y_2^4 + y_3^4$. The inverse system, call it M generated by these two elements of degree 4 will have $M_3 = \langle y_1^3, y_2^3, y_3^3 \rangle$, $M_2 = \langle y_1^2, y_2^2, y_3^2 \rangle$, $M_1 = \langle y_1, y_2, y_3 \rangle$, and $M_0 = \langle 1 \rangle$. So, if $I = \text{ann}_R(M)$ and $A = k[x_1, x_2, x_3]/I$ then the h -vector of A is $(1, 3, 3, 3, 2)$.

3. NON EXISTENCE OF A LEVEL O-SEQUENCE

First, we need the following lemma to prove the main theorem.

Lemma 3.1. *Let $R = k[x, y, z]$. If I is a homogeneous ideal of R such that R/I has the Hilbert function $\mathbf{H} = (1, 3, 4, 5, \dots)$, then two minimal generators in I in degree 2 has a linear common factor.*

Proof. If it were, let I be an Artinian ideal in $R = k[x, y, z]$ so that the Hilbert function of $A = R/I$ is

$$1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 4 \quad 2 \quad 0 \quad \rightarrow .$$

Let $\langle F, G \rangle = I_2$ and assume that F and G be a regular sequence. Since 4 and 5 have a maximal growth in degrees 2 and 3, we see that I does not have any generators in degree 3, that is,

$$\begin{aligned} R_1 I_2 &= I_3 \\ &= \langle Fx, Fy, Fz, Gx, Gy, Gz \rangle. \end{aligned}$$

Since $\dim_k I_3 = 5 = 10 - \mathbf{H}(R/I, 3)$, we have that one of Fx, Fy, Fz, Gx, Gy and Gz is a linear combination of the rest of 5 elements.

Without loss of generality, we may assume that

$$Fx \in \langle Fy, Fz, Gx, Gy, Gz \rangle.$$

Then

$$Fx = aFy + bFz + cGx + dGy + eGz$$

where $a, b, c, d, e \in k$, and so

$$F(x - ay - bz) = G(cx + dy + ez).$$

Since F and G be a regular sequence, we have that

$$F \mid (cx + dy + ez),$$

which is a contradiction, since $\deg F=2$. So F and G cannot be regular sequence, which means that they have a common factor, which has to be a linear since they are forms of degree 2. \square

Theorem 3.2. *The h -vector $\mathbf{H} = (1, 3, 4, 5, \dots, 6, 4, 2)$ is not a level h -vector.*

Proof. First we shall prove in the case of $\mathbf{H} = (1, 3, 4, 5, 6, 4, 2)$.

If it were, let I be an Artinian ideal in $R = k[x, y, z]$ so that the Hilbert function of $A = R/I$ is

$$1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 4 \quad 2 \quad 0 \quad \rightarrow$$

By Lemma 5, two forms in I_2 have to have a common factor.

Without loss generality, we may assume that $I_2 = \langle xy, xz \rangle$ or $\langle x^2, xy \rangle$.

Then $R_1 I_2 = \langle x^2 y, x^2 z, xy^2, xyz, xz^2 \rangle$ or $\langle x^3, x^2 y, xy^2, x^2 z, xyz \rangle$.

Thus

$$\mathbf{H}(R/R_1 I_2, 3) = \dim_k R_3 - \dim_k R_1 I_2 = 10 - 5 = 5 = \mathbf{H}(R/I, 3).$$

This means that $I_3 = R_1 I_2$.

Similarly, we have that

$$R_1 I_3 = \langle x^3 y, x^3 z, x^2 y^2, x^2 y z, x^2 z^2, x y^3, x y^2 z, x y z^2, x z^3 \rangle$$

or

$$\langle x^4, x^3 y, x^2 y^2, x^3 z, x^2 y z, x y^3, x y^2 z, x^2 z^2, x y z^2 \rangle.$$

Thus

$$\mathbf{H}(R/R_1 I_3, 4) = \dim_k R_4 - \dim_k (R_1 I_3) = 15 - 9 = 6 = \mathbf{H}(R/I, 4),$$

i. e., $I_4 = R_1 I_3$.

In other words, I has no generators in degree 3 and 4. I.e., we have that $(I_{\leq 4}) = (I_{\leq 2}) = (xy, xz)$ or (x^2, xy) .

Suppose $(I_{\leq 4}) = (xy, xz)$.

Let $S = k[x, y, z]$ and $F, G \in S_6$ be such that $\langle F, G \rangle^\perp = I$. Since $(I_{\leq 4}) = (xy, xz)$, we may assume that

$$F, G \in \langle Y^6, Y^5 Z, Y^4 Z^2, Y^3 Z^3, Y^2 Z^4, Y Z^5, Z^6, X^6 \rangle.$$

So, let

$$F = a_1 Y^6 + a_2 Y^5 Z + a_3 Y^4 Z^2 + a_4 Y^3 Z^3 + a_5 Y^2 Z^4 + a_6 Y Z^5 + a_7 Z^6 \\ + a X^6,$$

$$G = b_1 Y^6 + b_2 Y^5 Z + b_3 Y^4 Z^2 + b_4 Y^3 Z^3 + b_5 Y^2 Z^4 + b_6 Y Z^5 + b_7 Z^6.$$

If F and G are linear combination in $k[Y, Z]_6$, then the fifth contractions of F and G have at most two variables, that is, Y, Z . This means that $\mathbf{H}(A, 1) \leq 2$, which is a contradiction since $\mathbf{H}(A, 1) = 3$. Hence we cannot have both F and G in $k[Y, Z]_6$.

Without loss of generality, we may assume $a = 1$.

Now consider all the contractions of F and G . They may be viewed as vectors in k^7 as follows:

$$\begin{aligned} D_X(F) &= X^5 \\ &\leftrightarrow (0, 0, 0, 0, 0, 0, 1) \end{aligned}$$

$$\begin{aligned} D_Y(F) &= a_1Y^5 + a_2Y^4Z + a_3Y^3Z^2 + a_4Y^2Z^3 + a_5YZ^4 + a_6Z^5 \\ &\leftrightarrow (a_1, a_2, a_3, a_4, a_5, a_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(F) &= a_2Y^5 + a_3Y^4Z + a_4Y^3Z^2 + a_5Y^2Z^3 + a_6YZ^4 + a_7Z^5 \\ &\leftrightarrow (a_2, a_3, a_4, a_5, a_6, a_7, 0) \end{aligned}$$

$$\begin{aligned} D_X(G) &= 0 \\ &\leftrightarrow (0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_Y(G) &= b_1Y^5 + b_2Y^4Z + b_3Y^3Z^2 + b_4Y^2Z^3 + b_5YZ^4 + b_6Z^5 \\ &\leftrightarrow (b_1, b_2, b_3, b_4, b_5, b_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(G) &= b_2Y^5 + b_3Y^4Z + b_4Y^3Z^2 + b_5Y^2Z^3 + b_6YZ^4 + b_7Z^5 \\ &\leftrightarrow (b_2, b_3, b_4, b_5, b_6, b_7, 0). \end{aligned}$$

Since $\mathbf{H}(R/I, 5) = 4$, we have that a matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 \end{pmatrix}$$

has rank 4, and hence

$$\mathbf{A}_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{pmatrix}$$

has rank 3.

We now consider the double contractions of F and G , where we identify a polynomial

$$\alpha_1 Y^4 + \alpha_2 Y^3 Z + \alpha_3 Y^2 Z^2 + \alpha_4 Y Z^3 + \alpha_5 Z^4 + \beta X^4$$

as a vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta) \in k^6$. Then we obtain

$$\begin{aligned} D_{X,X}(F) &= X^4 \\ &\leftrightarrow (0, 0, 0, 0, 0, 1) \end{aligned}$$

$$\begin{aligned} D_{Y,Y}(F) &= a_1 Y^4 + a_2 Y^3 Z + a_3 Y^2 Z^2 + a_4 Y Z^3 + a_5 Z^4 \\ &\leftrightarrow (a_1, a_2, a_3, a_4, a_5, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Y}(F) &= a_2 Y^4 + a_3 Y^3 Z + a_4 Y^2 Z^2 + a_5 Y Z^3 + a_6 Z^4 \\ &\leftrightarrow (a_2, a_3, a_4, a_5, a_6, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Z}(F) &= a_3 Y^4 + a_4 Y^3 Z + a_5 Y^2 Z^2 + a_6 Y Z^3 + a_7 Z^4 \\ &\leftrightarrow (a_3, a_4, a_5, a_6, a_7, 0) \end{aligned}$$

$$\begin{aligned} D_{Y,Y}(G) &= b_1 Y^4 + b_2 Y^3 Z + b_3 Y^2 Z^2 + b_4 Y Z^3 + b_5 Z^4 \\ &\leftrightarrow (b_1, b_2, b_3, b_4, b_5, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Y}(G) &= b_2 Y^4 + b_3 Y^3 Z + b_4 Y^2 Z^2 + b_5 Y Z^3 + b_6 Z^4 \\ &\leftrightarrow (b_2, b_3, b_4, b_5, b_6, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Z}(G) &= b_3 Y^4 + b_4 Y^3 Z + b_5 Y^2 Z^2 + b_6 Y Z^3 + b_7 Z^4 \\ &\leftrightarrow (b_3, b_4, b_5, b_6, b_7, 0) \end{aligned}$$

Since $\mathbf{H}(R/I, 4) = 6$, we see that a matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_3 & b_4 & b_5 & b_6 & b_7 & 0 \end{pmatrix}$$

has rank 6, and so

$$\mathbf{B}_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ b_2 & b_3 & b_4 & b_5 & b_6 \\ b_3 & b_4 & b_5 & b_6 & b_7 \end{pmatrix}$$

has rank 5.

Since the rank of \mathbf{A}_1 is 3, one of four row vectors of \mathbf{A}_1 has to be a linear combination of the other three.

Case 1–1.

The row space of

$$\mathbf{A}_1 = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle.$$

Since

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5, a_6) \\ & \in \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle, \end{aligned}$$

we have

$$(a_1, a_2, a_3, a_4, a_5) \in \langle (a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle$$

and

$$(a_2, a_3, a_4, a_5, a_6) \in \langle (a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle.$$

Then we have

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5) \\ & \in \langle (a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle, \end{aligned}$$

which means that the matrix \mathbf{B}_1 can have rank at most 4, a contradiction.

Case 1–2. The row space of

$$\mathbf{A}_1 = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle$$

Since

$$\begin{aligned} & (a_2, a_3, a_4, a_5, a_6, a_7) \\ & \in \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle, \end{aligned}$$

we have

$$(a_2, a_3, a_4, a_5, a_6) \in \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle$$

and

$$(a_3, a_4, a_5, a_6, a_7) \in \langle (a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle.$$

Then we have

$$\begin{aligned} & (a_3, a_4, a_5, a_6, a_7) \\ & \in \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle, \end{aligned}$$

which means that the matrix \mathbf{B}_1 can have rank at most 4, a contradiction.

Case 1–3. The row space of

$$\mathbf{A}_1 = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (a_2, a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle.$$

Since

$$\begin{aligned} & (b_1, b_2, b_3, b_4, b_5, b_6) \\ & \in \langle (a_1, a_2, a_3, a_4, a_5, a_6), (a_2, a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle, \end{aligned}$$

We have

$$(b_1, b_2, b_3, b_4, b_5) \in \langle (a_1, a_2, a_3, a_4, a_5), (a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6) \rangle$$

and

$$(b_2, b_3, b_4, b_5, b_6) \in \langle (a_2, a_3, a_4, a_5, a_6), (a_3, a_4, a_5, a_6, a_7), (b_3, b_4, b_5, b_6, b_7) \rangle,$$

Then we have

$$\begin{aligned} & (b_1, b_2, b_3, b_4, b_5) \\ & \in \langle (a_1, a_2, a_3, a_4, a_5), (a_2, a_3, a_4, a_5, a_6), (a_3, a_4, a_5, a_6, a_7), (b_3, b_4, b_5, b_6, b_7) \rangle, \end{aligned}$$

which means that the matrix \mathbf{B}_1 can have rank at most 4, a contradiction.

Case 1–4. The row space of

$$\mathbf{A}_1 = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6) \rangle$$

Since

$$\begin{aligned} & (b_2, b_3, b_4, b_5, b_6, b_7) \\ & \in \langle (a_1, a_2, a_3, a_4, a_5, a_6), (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6) \rangle, \end{aligned}$$

we have

$$(b_2, b_3, b_4, b_5, b_6) \in \langle (a_1, a_2, a_3, a_4, a_5), (a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5) \rangle$$

and

$$(b_3, b_4, b_5, b_6, b_7) \in \langle (a_2, a_3, a_4, a_5, a_6), (a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6) \rangle.$$

Then we have

$$(b_3, b_4, b_5, b_6, b_7) \in \langle (a_1, a_2, a_3, a_4, a_5), (a_2, a_3, a_4, a_5, a_6), (a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5) \rangle,$$

which means that the matrix \mathbf{B}_1 can have rank at most 4, a contradiction.

By **Case 1-1~1-4**, \mathbf{B}_1 cannot have rank 5.

Hence, the O-sequence $\mathbf{H} = (1, 3, 4, 5, 6, 4, 2)$ is not level.

Now consider the case of $(I_{\leq 4}) = (x^2, xy)$.

Let $S = k[x, y, z]$ and $F, G \in S_6$ be such that $\langle F, G \rangle^\perp = I$. Since $(I_{\leq 4}) = (x^2, xy)$, we may assume that

$$F, G \in \langle Y^6, Y^5Z, Y^4Z^2, Y^3Z^3, Y^2Z^4, YZ^5, Z^6, XZ^5 \rangle.$$

So, let

$$F = a_1Y^6 + a_2Y^5Z + a_3Y^4Z^2 + a_4Y^3Z^3 + a_5Y^2Z^4 + a_6YZ^5 + a_7Z^6 + aXZ^5,$$

$$G = b_1Y^6 + b_2Y^5Z + b_3Y^4Z^2 + b_4Y^3Z^3 + b_5Y^2Z^4 + b_6YZ^5 + b_7Z^6.$$

Since $\mathbf{H}(A, 1) = 3$ we cannot have both F and G in $k[Y, Z]_6$ and so we may assume $a = 1$

Now consider all the contractions of F and G . They may be viewed as vectors in k^7 as follows :

$$\begin{aligned} D_X(F) &= Z^5 \\ &\leftrightarrow (0, 0, 0, 0, 0, 1, 0) \end{aligned}$$

$$\begin{aligned} D_Y(F) &= a_1Y^5 + a_2Y^4Z + a_3Y^3Z^2 + a_4Y^2Z^3 + a_5YZ^4 + a_6Z^5 \\ &\leftrightarrow (a_1, a_2, a_3, a_4, a_5, a_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(F) &= a_2Y^5 + a_3Y^4Z + a_4Y^3Z^2 + a_5Y^2Z^3 + a_6YZ^4 + a_7Z^5 + XZ^4 \\ &\leftrightarrow (a_2, a_3, a_4, a_5, a_6, a_7, 1) \end{aligned}$$

$$\begin{aligned} D_X(G) &= 0 \\ &\leftrightarrow (0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_Y(G) &= b_1Y^5 + b_2Y^4Z + b_3Y^3Z^2 + b_4Y^2Z^3 + b_5YZ^4 + b_6Z^5 \\ &\leftrightarrow (b_1, b_2, b_3, b_4, b_5, b_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(G) &= b_2Y^5 + b_3Y^4Z + b_4Y^3Z^2 + b_5Y^2Z^3 + b_6YZ^4 + b_7Z^5 \\ &\leftrightarrow (b_2, b_3, b_4, b_5, b_6, b_7, 0) \end{aligned}$$

Hence

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 1 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 \end{pmatrix}$$

has rank 4 since $\mathbf{H}(R/I, 5) = 4$, and hence

$$\mathbf{A}_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

has rank 2.

We now consider the double contractions of F and G , where we identify a polynomial

$$\alpha_1 Y^4 + \alpha_2 Y^3 Z + \alpha_3 Y^2 Z^2 + \alpha_4 Y Z^3 + \alpha_5 Z^4 + \beta X Z^3$$

as a vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta) \in k^6$. Then we obtain

$$\begin{aligned} D_{Z,X}(F) &= Z^4 \\ &\leftrightarrow (0, 0, 0, 0, 1, 0) \end{aligned}$$

$$\begin{aligned} D_{Y,Y}(F) &= a_1 Y^4 + a_2 Y^3 Z + a_3 Y^2 Z^2 + a_4 Y Z^3 + a_5 Z^4 \\ &\leftrightarrow (a_1, a_2, a_3, a_4, a_5, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Y}(F) &= a_2 Y^4 + a_3 Y^3 Z + a_4 Y^2 Z^2 + a_5 Y Z^3 + a_6 Z^4 \\ &\leftrightarrow (a_2, a_3, a_4, a_5, a_6, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Z}(F) &= a_3 Y^4 + a_4 Y^3 Z + a_5 Y^2 Z^2 + a_6 Y Z^3 + a_7 Z^4 + X Z^3 \\ &\leftrightarrow (a_3, a_4, a_5, a_6, a_7, 1) \end{aligned}$$

$$\begin{aligned} D_{Y,Y}(G) &= b_1 Y^4 + b_2 Y^3 Z + b_3 Y^2 Z^2 + b_4 Y Z^3 + b_5 Z^4 \\ &\leftrightarrow (b_1, b_2, b_3, b_4, b_5, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Y}(G) &= b_2 Y^4 + b_3 Y^3 Z + b_4 Y^2 Z^2 + b_5 Y Z^3 + b_6 Z^4 \\ &\leftrightarrow (b_2, b_3, b_4, b_5, b_6, 0) \end{aligned}$$

$$\begin{aligned} D_{Z,Z}(G) &= b_3 Y^4 + b_4 Y^3 Z + b_5 Y^2 Z^2 + b_6 Y Z^3 + b_7 Z^4 \\ &\leftrightarrow (b_3, b_4, b_5, b_6, b_7, 0) \end{aligned}$$

Hence we have

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 & 1 \\ b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_3 & b_4 & b_5 & b_6 & b_7 & 0 \end{pmatrix}$$

has rank 6 since $\mathbf{H}(R/I, 4) = 6$, and so

$$\mathbf{B}_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 \\ b_2 & b_3 & b_4 & b_5 \\ b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

has rank 4. Since the rank of \mathbf{A}_1 is 2, one of three row vectors of \mathbf{A}_1 has to be a linear combination of the other two.

Case 2–1.

The row space of

$$\mathbf{A}_1 = \langle (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle.$$

Since

$$(a_1, a_2, a_3, a_4, a_5) \in \langle (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle,$$

we have that

$$(a_1, a_2, a_3, a_4) \in \langle (b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5) \rangle$$

and

$$(a_2, a_3, a_4, a_5) \in \langle (b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6) \rangle.$$

Then we obtain to

$$(a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5) \in \langle (b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6) \rangle,$$

which means that the matrix \mathbf{B}_1 can have rank at most 3, a contradiction.

Case 2-2.

The row space of

$$\mathbf{A}_1 = \langle (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6) \rangle.$$

Since

$$(b_1, b_2, b_3, b_4, b_5) \in \langle (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6) \rangle$$

we have that

$$(b_1, b_2, b_3, b_4) \in \langle (a_1, a_2, a_3, a_4), (b_2, b_3, b_4, b_5) \rangle$$

and

$$(b_2, b_3, b_4, b_5) \in \langle (a_2, a_3, a_4, a_5), (b_3, b_4, b_5, b_6) \rangle.$$

Then we obtain to

$$(b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5) \in \langle (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (b_3, b_4, b_5, b_6) \rangle,$$

which means that the matrix \mathbf{B}_1 can have rank at most 3, a contradiction.

Case 2-3.

The row space of

$$\mathbf{A}_1 = \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5) \rangle.$$

Since

$$(b_2, b_3, b_4, b_5, b_6) \in \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5) \rangle,$$

we have that

$$(b_2, b_3, b_4, b_5) \in \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$$

and

$$(b_3, b_4, b_5, b_6) \in \langle (a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5) \rangle.$$

Then we obtain to

$$(b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6) \in \langle (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4) \rangle,$$

which means that the matrix \mathbf{B}_1 can have rank at most 3, a contradiction.

By **Case 2-1** \sim **2-3**, \mathbf{B}_1 cannot have rank 4.

Hence, the O -sequence $\mathbf{H} = (1, 3, 4, 5, 6, 4, 2)$ is not level.

Using the same ideas as above, we can show that any O -sequence of the form

$$h = (1, 3, 4, 5, \dots, 6, 4, 2)$$

is not level. □

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ABSTRACT

THE EXAMPLE OF A NON LEVEL O-SEQUENCE OF CODIMENSION 3 ENDING 6, 4, 2.

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Supervised by professor Shin Yong Su Ph.D.

A. V. Geramita, T. Harima, J. C. Migliore and Y. S. Shin proved that if a level O-sequence of codimension 3 of the form

$$1, 3, \dots, 6, 3, 2$$

or

$$1, 3, \dots, 5, 3, 2$$

are not level in [7].

In this paper, we proved a level O-sequence of codimension 3

$$1, 3, 4, 5, \dots, 6, 4, 2$$

is not level using Inverse System or Macaulay duality.