1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be a n-variable polynomial ring over an algebraically closed field $k = \bar{k}$ and I be a homogeneous ideal of R. Since R_i is the vector space of dimension $\binom{i+n-1}{n-1}$ generated by all the monomials in R having degree i and $I = \oplus_{i=0}^{\infty} I_i$, we get that

$$A = R/I = \oplus_{i=0}^{\infty}(R_i/I_i) = \oplus_{i=0}^{\infty} A_i$$

is a graded ring. The numerical function

$$\mathbf{H}_A(t) := \dim_k A_t = \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the ring A (or of the ideal I). If $\mathbb{X}$ is a finite set of distinct points in $\mathbb{P}^n$ and $A = R/I_{\mathbb{X}}$ is a coordinate ring of $\mathbb{X}$, then we sometimes denote the Hilbert function of A by $\mathbf{H}_{\mathbb{X}}(t) := \mathbf{H}_A(t)$.

Let h and i be positive integers. Then h can be written uniquely in the form

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expansion for h is called the *i-binomial expansion* of h. Also, define

$$h^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_{i-1} + 1}{(i-1)+1} + \cdots + \binom{m_j + 1}{j+1},$$

and $0^{\langle i \rangle} = 0$. Let $\mathbf{T} = (h_0, h_1, \ldots, h_i, \ldots)$ be a sequence of non-negative integers. We say that $\mathbf{T}$ is an *O-sequence* if $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i \rangle}$

for all $i \geq 1$. Given an O-sequence $\mathbf{T} = (h_0, h_1, \ldots, h_i, \ldots)$, we can differentiate it to get a new sequence

$$\Delta\mathbf{T} = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \ldots)$$

and we call $\Delta\mathbf{T}$ the *first difference* of $\mathbf{T}$. If $\Delta\mathbf{T}$ is an O-sequence again, $\mathbf{T}$ is called a *differentiable O-sequence*.

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R. The *h-vector* of A is $h(A) = (h_0, h_1, \ldots, h_\ell)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and ℓ is the last index such that $\dim_k A_k \neq 0$. We call ℓ the *socle degree* A. Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the *codimension* of A.

The socle of A is the annihilator of the maximal homogeneous ideal $\overline{m} = (\bar{x}_1, \ldots, \bar{x}_n)$, that is,

$$\operatorname{soc}(A) = \{a \in A \mid a \cdot \overline{m} = 0\}.$$

Let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to R \to R/I \to 0$$

be a minimal free resolution of R/I. Then A is a *level algebra of type* m if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. In particular, if $m = 1$, then we say that A is a *Gorenstein algebra* and the h-vector of A is a *Gorenstein sequence*. It is well-known that an Artinian graded algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_\ell$ is a level if $\operatorname{soc}(A) = A_\ell$.

Now we recall the main facts of the theory of *inverse systems*, or *Macaulay duality* (see [6], [9]). Let $S = k[y_1, \ldots, y_n]$, and consider S as a graded R-module where the action of x_i on S is a partial differentiation with respect to y_i. There is a one-to-one correspondence between graded algebras R/I and finitely generated R-submodules M of S, where $I = \mathrm{Ann}(M)$ is the annihilator of M in R and, conversely, $M = I^{-1}$ is the R-submodule of S which is annihilated by I.

For graded Artinian level algebras, it has been recently studied (see [2], [3], [4], [5], [7], [8], [10], [12], [13]).

In [12], F. Zanello found how to produce a non-unimodal level O-sequence of codimension 3

$$1, \quad 3, \quad \cdots \quad t, \quad t, \quad t+1, \quad t, \quad t, \quad t+1, \quad \cdots \quad t, \quad t, \quad t+1$$

having local maxima as many times as we want.

In this thesis, using a different way from Zanello's construction, we find how to produce a level O-sequence of codimension 3 having local maxima as many times as we desire (see Theorem 3.2). Moreover, we prove that the O-sequence we constructed is symmetric between any two local maxima (see Theorem 3.5).

A computer program CoCoA was used for all examples in this thesis with algorithms which were provided by Professor Shin (see Algorithms 2.1, 2.3, and 2.6).

2. SOME ALGORITHMS FOR OBTAINING LEVEL O-SEQUENCES

In this section, we shall introduce some algorithms to obtain level O-sequence easily using Theorem 3.1 from CoCoA [11]. Moreover, we shall discuss the properties of non-unimodal O-sequences.

First of all, the following algorithm is to check if the given sequence is an O-sequence.

Algorithm 2.1 (CoCoA, Checking O-sequences)**.**

```
Define CHECKOSequence(T)
  A1:="==> Yes, this is an O-Sequence.";
  A2:="==> No, this is NOT an O-Sequence." ;
  A:=A1;
  For I:= 2 To Len(T)-1 Do
      J:=I+1;
     S1:=Comp(T,I);
     S2:=Comp(T,J);
          BinValue:=BinExp(S1,I-1,1,1);
  --Print I, J, S2, BinValue, NewLine;
  If BinValue < S2 Then A:= A2
  EndIf;
  EndFor;
  S:=Comp(T,1);
  If S > 1 Then Print "The 1st component should be 1,
           so this is NOT an O-Sequence.", NewLine
  EndIf;
  If S < 1 Then Print "The 1st component should be 1,
           so this is NOT an O-Sequence.", NewLine
  EndIf;
  If S=1 Then  Print T, A
  EndIf;
EndDefine;
```

Example 2.2 (CoCoA)**.** If we run `CHECKOSequence(T)` from CoCoA, then we can check if a given sequence is an O-sequence as follows:

```
CHECKOSequence([1,4,7,10,13,6,3]);
[1, 4, 7, 10, 13, 6, 3]==> Yes, this is an O-Sequence.
-------------------------------------------------------

CHECKOSequence([1,4,7,10,13,17]);
[1, 4, 7, 10, 13, 17]==> No, this is NOT an O-Sequence.
-------------------------------------------------------
```

Hence, the first sequence $(1, 4, 7, 10, 13, 6, 3)$ is an O-sequence, but the second sequence $(1, 4, 7, 10, 13, 17)$ is not an O-sequence.

Here we need the following algorithm, which we can obtain the differentiable O-sequence from an O-sequence made by Algorithm 2.1.

Algorithm 2.3 (CoCoA: Adding Up O-sequences)**.**

```
Define ADDUPHilbert(L)
   S:=[];
    For I := 2 To Len(L)
      Do S1:=Sum(First(L,I));
      Append(S,S1);
    EndFor;
   S2:=Comp(S,Len(S));
   S:=[S];
   Append(S,S2);
   S;
EndDefine;
```

Example 2.4 (CoCoA)**.** If we also run `ADDUPHilbert(L)` from CoCoA, then we can add up if a given O-sequence as follows:

```
ADDUPHilbert([1, 4, 7, 10, 13, 6, 3]);
[[5, 12, 22, 35, 41, 44], 44]
-------------------------------------
```

So, we can obtain the differentiable O-sequence $(1, 5, 12, 22, 35, 41, 44)$.

In [1], the following interesting O-sequence of codimension 3 was studied:

$$\mathbf{H} \; : \; h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad \cdots \quad \overset{(d+s-1)\text{-st}}{h_d} \quad h_{d+s} \tag{2.1}$$

where $s \geq 2$ and $h_d < h_{d+s}$. Moreover, the following was also proved: some graded algebra with Hilbert function $\mathbf{H}$ of codimension 3 in equation (2.1) has a socle element in degree $d + s - 2$, and hence cannot be a level. So we have the following natural question here.

Question 2.5. Let $\mathbf{H}$ be an O-sequence in equation (2.1). Is $\mathbf{H}$ a non-level O-sequence?

Before we make an example, we introduce the following algorithm which we can obtain a level O-sequence using CoCoA based on Theorem 3.1

Algorithm 2.6 (CoCoA: Obtaining Level h-vector)**.**

```
Define LEVELHVECTOR(T)
  NewT:=[1];
  R:=Comp(T,2);
```

```
  E:=Len(T)-1;
  For J := 2 To Len(T) Do
   I:=J-1;
          Ti:=Comp(T,J);
   T1:=Bin(R-1+E-I,E-I);
   T2:=Bin(R-1+I,I);
     NewTi:=Min(Ti+T1,T2);
    Append(NewT,NewTi);
  EndFor;
  Print "From h=", T, "   and r=", R, NewLine;
  Print "We have T=", NewT
EndDefine;
```

Example 2.7 (CoCoA)**.** Consider a level h-vector $(1, 5, 12, 22, 35, 41, 44)$ in Example 2.4. Apply Algorithm 2.6, we have another level O-sequence as follows:

```
LEVELHVECTOR([1, 5, 12, 22, 35, 41, 44]);
From h=[1, 5, 12, 22, 35, 41, 44]   and r=5
We have T=[1, 5, 15, 35, 50, 46, 45]
-------------------------------------------
```

So, we can obtain another level O-sequence $(1, 5, 15, 35, 50, 46, 45)$.

3. THE CONSTRUCTION OF A NON-UNIMODAL LEVEL O-SEQUENCE OF CODIMENSION 3

Theorem 3.1 (Theorem 4.8A, [9]). *Let* $h = (h_0, h_1, \ldots, h_e)$ *be the* h*-vector of a level algebra* $A = R/\mathrm{Ann}(M)$ *where* $R = k[x_1, \ldots, x_r]$*. Then, if* F *is a generic form of degree* e*, the level algebra* $A = R/\mathrm{Ann}(\langle M, F\rangle)$ *has* h*-vector* $\mathbf{H} = (H_0, H_1, \ldots, H_e)$ *where, for* $i = 1, \ldots, e$*,*

$$H_i = \min\left\{ h_i + \binom{r-1+e-i}{e-i}, \binom{r-1+i}{i} \right\}.$$

Theorem 3.2. *Let* $\mathbf{H}'$ *be a Hilbert function such that*

$$\begin{aligned}\boldsymbol{\Delta}\mathbf{H}' = \ & (1,2,3,...,s+1,\\ & \underbrace{3^{N+1},...,3^{N+1}}_{3^{N+1}\text{-times}}, \underbrace{3^{N},...,3^{N}}_{3^{N}\text{-times}},\\ & \qquad\vdots\\ & \underbrace{3^{3},...,3^{3}}_{3^{3}\text{-times}}, \underbrace{3^{2},...,3^{2}}_{3^{2}\text{-times}}, \underbrace{3,3,3}_{3\text{-times}})\end{aligned}$$

where $s \gg 0$*, and let* $\mathbf{H}$ *be in Theorem 3.1. Then* $\mathbf{H}$ *is a level O-sequence of codimension* 3 *having the same* N*-local maxima.*

Proof. We shall prove that all N-local maxima are the same. In other words, we shall show that

$$H_{s+3^{N+1}+3^{N}+\cdots+3^{N-\ell+1}} = H_{s+3^{N+1}+3^{N}+\cdots+3^{N-\ell}}.$$

for every $\ell = 0, 1, \cdots, N-1$.

In fact, by Theorem 3.1

$$\begin{aligned} H_{s+3^{N+1}+3^{N}+\cdots+3^{N-\ell+1}} &= \binom{s+2}{2} + \underbrace{3^{N+1}+\cdots+3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &\quad + \underbrace{3^{N-\ell+1}+\cdots+3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \binom{\frac{3^{N-\ell+1}}{2}+\frac{1}{2}}{\frac{3^{N-\ell+1}}{2}-\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} H_{s+3^{N+1}+3^{N}+\cdots+3^{N-\ell}} &= \binom{s+2}{2} + \underbrace{3^{N+1}+\cdots+3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &\quad + \underbrace{3^{N-\ell}+\cdots+3^{N-\ell}}_{3^{N-\ell}\text{-times}} + \binom{\frac{3^{N-\ell}}{2}+\frac{1}{2}}{\frac{3^{N-\ell}}{2}-\frac{3}{2}}, \end{aligned}$$

and so,

$$\begin{aligned}
& H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell+1}} - H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell}} \\
= \ & \binom{\frac{3^{N-\ell+1}}{2}+\frac{1}{2}}{\frac{3^{N-\ell+1}}{2}-\frac{3}{2}} - 3^{2N-2\ell} - \binom{\frac{3^{N-\ell}}{2}+\frac{1}{2}}{\frac{3^{N-\ell}}{2}-\frac{3}{2}} \\
= \ & \frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{1-\ell+N}\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{1-\ell+N}\right) - 3^{2N-2\ell} \\
& -\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{-\ell+N}\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{-\ell+N}\right) \\
= \ & \left(-\frac{1}{8}+\frac{3^{2-2\ell+2N}}{8}\right) - 3^{2N-2\ell} - \left(-\frac{1}{8}+\frac{3^{-2\ell+2N}}{8}\right) \\
= \ & 0,
\end{aligned}$$

which follows

$$H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell+1}} = H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell}}$$

for every $\ell = 0, 1, \cdots, N-1,$ as we wished. $\square$

Example 3.3 (CoCoA). Let $\mathbf{H}'$ be an O-sequence of codimension 3 such that

$$\mathbf{\Delta H}' = \ (1,2,3,...,200,\underbrace{81,\cdots,81}_{\text{81-times}},\underbrace{27,\cdots,27}_{\text{27-times}},\underbrace{9,\cdots,9}_{\text{9-times}},\underbrace{3,3,3}_{\text{3-times}}).$$

Then using Thoerem 3.1 and Algorithm 2.6, we obtain another level O-sequence as follows:

$$\begin{aligned}
& \mathbf{H} \\
= \ & (1,3,6,\cdots,27402,27441, \\
& \ \mathbf{27481},27468,27456,27445,27435,27426,27418,27411,27405, \\
& \ 27400,27396,27393,27391,27390,27390,27391,27393,27396, \\
& \ 27400,27405,27411,27418,27426,27435,27445,27456,27468, \\
& \ \mathbf{27481},27477,27474,27472,27471,27471,27472,27474,27477, \\
& \ \mathbf{27481},27480,27480,27481)
\end{aligned}$$

.

In Example 3.3, we find an interesting result, that is, the Hilbert function is symmetric between any two local maxima. Hence we have a natural question here as follows.

Question 3.4. Let $\mathbf{H}$ be as in Thoerem 3.2. Is the Hilbert function $\mathbf{H}$ symmetric between any two local maxima?

We have an answer to Question 3.4 in Theorem 3.5.

Theorem 3.5. *Let* $\mathbf{H}$ *be as in Question 3.4. Then the Hilbert function* $\mathbf{H}$ *is symmetric between two local maxima.*

In other words,

$$H_{s+3^{N+1}+\cdots+3^{N-\ell+1}+k} = H_{s+3^{N+1}+\cdots+3^{N-\ell+1}+3^{N-\ell}-k}$$

where for $\ell = 0, 1, \cdots, N-1$ *and* $k \leq \frac{\left(3^{N+1}+\cdots+3^{N-\ell+1}\right)+\left(3^{N+1}+\cdots+3^{N-\ell+1}+3^{N-\ell}\right)}{2}$.

Proof. Let ℓ and k be as above,

$$\begin{aligned} H_{s+3^{N+1}+\cdots+3^{N-\ell+1}+k} &= \binom{s+2}{2} + \underbrace{3^{N+1}+\cdots+3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &\quad + \underbrace{3^{N-\ell+1}+\cdots+3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \underbrace{3^{N-\ell}+\cdots+3^{N-\ell}}_{k\text{-times}} \\ &\quad + \binom{\frac{3^{N-\ell+1}}{2}+\frac{1}{2}-k}{\frac{3^{N-\ell+1}}{2}-\frac{3}{2}-k}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} H_{s+3^{N+1}+\cdots+3^{N-\ell}-k} &= \binom{s+2}{2} + \underbrace{3^{N+1}+\cdots+3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &\quad + \underbrace{3^{N-\ell+1}+\cdots+3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \underbrace{3^{N-\ell}+\cdots+3^{N-\ell}}_{3^{N-\ell}-k\text{-times}} \\ &\quad + \binom{\frac{3^{N-\ell}}{2}+\frac{1}{2}+k}{\frac{3^{N-\ell}}{2}-\frac{3}{2}+k} \end{aligned}$$

hence we have

$$
\begin{aligned}
& H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell+1}+k} - H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell}-k} \\
= \quad & \underbrace{3^{N-\ell}+\cdots+3^{N-\ell}}_{k\text{-times}} + \binom{\frac{3^{N-\ell+1}}{2}+\frac{1}{2}-k}{\frac{3^{N-\ell+1}}{2}-\frac{3}{2}-k} \\
& -\underbrace{\left(3^{N-\ell}+\cdots+3^{N-\ell}\right)}_{3^{N-\ell}-k\text{-times}} - \binom{\frac{3^{N-\ell}}{2}+\frac{1}{2}+k}{\frac{3^{N-\ell}}{2}-\frac{3}{2}+k} \\
= \quad & \left(2k-3^{-\ell+N}\right)3^{-\ell+N} \\
& +\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{1-\ell+N}-k\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{1-\ell+N}-k\right) \\
& -\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{-\ell+N}+k\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{-\ell+N}+k\right) \\
= \quad & \left(2k-3^{-\ell+N}\right)3^{-\ell+N} + \left(-\frac{1}{8}+\frac{3^{2-2\ell+2N}}{8}-\frac{3^{1-\ell+N}}{2}k+\frac{k^2}{2}\right) \\
& -\left(-\frac{1}{8}+\frac{3^{-2\ell+2N}}{8}+\frac{3^{-\ell+N}}{2}k+\frac{k^2}{2}\right) \\
= \quad & 0,
\end{aligned}
$$

which means that

$$
H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell+1}+k} = H_{s+3^{N+1}+3^N+\cdots+3^{N-\ell}-k}
$$

for such ℓ and k, as we desired. $\square$

We also find a very interesting level O-sequence in Example 3.6.

Example 3.6. [CoCoA] Consider an O-sequence $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 3, 3, 3, 3)$. Using Algorithm 2.3, we obtain a level O-sequence of codimension 3 as follows:

$$\mathbf{H} = (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 69, 72, 75, 78).$$

Then by Thoerem 3.1 and Algorithm 2.6 we have another level O-sequence of codimension 3

$$\mathbf{T} = (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \mathbf{66}, \mathbf{78}, \mathbf{78}, \mathbf{78}, \mathbf{79}),$$

which has 78 three times in degrees 11, 12 and 13. Unfortunately, we don't have any other example of a level O-sequence of codimension 3 of type $\mathbf{T}$ as above. In other words, $\mathbf{H}(d-1) = \mathbf{H}(d) = \mathbf{H}(d+1) < \mathbf{H}(d+2)$ for some $d > 0$.

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