申龍秀 教授指導
碩士學位 請求論文

THE BEHAVIOR OF LOCAL MINIMA OF A LEVEL O-SEQUENCE OF CODIMENSION 3 HAVING THE SAME LOCAL MAXIMA

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誠信女子大學校 教育大學院
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認 准 書

柳 奈 影의 碩士學位 論文을 認准함

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목 차

논 문 개 요

[12]에서 F.Zanello는 여차원이 3인 단일 양상이 아닌 Level O-수열(Non-Unimodal Level O-sequence)에 관하여 다음과 같은 Zanello의 구조를 발견하였는데, (1,3,...t,t,t+1,t,t,t+1,...t,t,t+1) 이 구조는 원하는 만큼의 극대값(Maxima)을 가진다 .

[9]에서 Zanello의 구조를 다양하게 사용하여, 새로운 여차원이 3인 단일 양상이 아닌 Level O-수열을 발견하였으며, 이 Level O-수열이 동일한 극대값을 가지며, 또한 이 Level O-수열의 어떠한 두 개의 극대값 사이가 항상 대칭적(Symmetric)임을 증명하였다.

우리는 이 논문에서 Kim이 발견한 새로운 Level O-수열을 사용하여, 이 Level O-수열의 극소값은 두 개의 연속한 차수에서 가지는 것을 증명하였다. 또한 이 Level O-수열의 극소값은 증가(increasing)함을 증명하였다.

이 논문에서 모든 예제는 컴퓨터 프로그램 CoCoA의 알고리즘이 사용되었다.

1. Introduction

Let $R = k[x_1, \dots, x_n]$ be a n-variable polynomial ring over an algebraically closed field $k = \bar{k}$ and I be a homogeneous ideal of R. Since R_i is the vector space of dimension $\binom{i+n-1}{n-1}$ generated by all the monomials in R having degree i and $I = \oplus_{i=0}^{\infty} I_i$, we get that

$$A = R/I = \oplus_{i=0}^{\infty}(R_i/I_i) = \oplus_{i=0}^{\infty} A_i$$

is a graded ring. The numerical function

$$\mathbf{H}_A(t) := \dim_k A_t = \dim_k R_t - \dim_k I_t$$

is called the ***Hilbert function*** of the ring A (or of the ideal I). Let h and i be positive integers. Then h can be written uniquely in the form

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expansion for h is called the ***i-binomial expansion*** of h. Also, define

$$h^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_{i-1}+1}{(i-1)+1} + \cdots + \binom{m_j+1}{j+1},$$

and $0^{\langle i \rangle} = 0$. Let $\mathbf{T} = (h_0, h_1, \dots, h_i, \dots)$ be a sequence of non-negative integers. We say that $\mathbf{T}$ is an ***O-sequence*** if $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i \rangle}$ for all $i \geq 1$. Given an O-sequence $\mathbf{T} = (h_0, h_1, \dots, h_i, \dots)$, we can differentiate it to get a new sequence

$$\Delta\mathbf{T} = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \dots)$$

and we call $\Delta\mathbf{T}$ the ***first difference*** of $\mathbf{T}$. If $\mathbf{T}$ is an O-sequence again, $\mathbf{T}$ is called a ***differentiable O-sequence***.

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R. The ***h-vector*** of A of $h(A) = (h_0, h_1, \ldots, h_\ell)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and ℓ is the last index such that $\dim_k A_k \neq 0$. We call ℓ the ***socle degree*** A. Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the ***codimension*** of A.

The socle of A is the annihilator of the maximal homogeneous ideal $\overline{m} = (\bar{x}_1, \ldots, \bar{x}_n)$, that is,

$$\mathrm{soc}(A) = \{a \in A \mid a \cdot \overline{m} = 0\}.$$

Let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to R \to R/I \to 0$$

be a minimal free resolution of R/I. Then A is a ***level algebra of type*** m if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. In particular, if $m = 1$, then we say that A is a ***Gorenstein algebra*** and the h-vector of A is a ***Gorenstein sequence***. It is well-known that an Artinian graded algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_\ell$ is **level** if $\mathrm{soc}(A) = A_\ell$.

For graded Artinian level algebras, it has been recently studied (see [1] - [5], [7] - [11], [13], [14]).

In [13], F. Zanello found how to produce a non-unimodal level O-sequence of codimension 3

$$1 \quad 3 \quad \cdots \quad t \quad t \quad t+1 \quad t \quad t \quad t+1 \quad \cdots \quad t \quad t \quad t+1$$

having local maxima as many times as we want.

In [9], using a different idea from Zanello's construction, she found how to produce a level O-sequence of codimension 3 having local maxima as many times as we desire . She also proved that the O-sequence we constructed is symmetric between any two local maxima.

Let **H** be a non-unimodal level O-sequence which Kim produced in [9]. In this thesis, we show that any local minima of **H** exist in two consecutive degrees (see Theorem 3.4). Moreover, we prove that those local minima are increasing (see Theorem 3.5).

A computer program CoCoA [12] was used for all examples in this thesis with algorithms which were provided by Professor Shin (see Algorithms 2.1, 2.3, and 2.6).

2. Some Algorithms for Obtaining Level O-sequences

In this section, we shall introduce some algorithms to obtain level O-sequence easily using Theorem 2.5 from CoCoA [12]. Moreover, we shall discuss the properties of non-unimodal O-sequences.

First of all, the following algorithm is to check if the given sequence is an O-sequence.

Algorithm 2.1 (CoCoA, Checking O-sequences)**.**

```
Define CHECKOSequence(T)
  A1:="==> Yes, this is an O-Sequence.";
  A2:="==> No, this is NOT an O-Sequence." ;
  A:=A1;
  For I:= 2 To Len(T)-1 Do
      J:=I+1;
     S1:=Comp(T,I);
     S2:=Comp(T,J);
          BinValue:=BinExp(S1,I-1,1,1);
  --Print I, J, S2, BinValue, NewLine;
  If BinValue < S2 Then A:= A2
  EndIf;
  EndFor;
  S:=Comp(T,1);
  If S > 1 Then Print "The 1st component should be 1,
           so this is NOT an O-Sequence.", NewLine
  EndIf;
  If S < 1 Then Print "The 1st component should be 1,
           so this is NOT an O-Sequence.", NewLine
  EndIf;
  If S=1 Then  Print T, A
  EndIf;
EndDefine;
```

Example 2.2 (CoCoA)**.** If we run `CHECKOSequence(T)` from CoCoA, then we can check if a given sequence is an O-sequence as follows:

```
CHECKOSequence([1,3,5,7,9,11,4,2]);
[1, 3, 5, 7, 9, 11, 4, 2]==> Yes, this is an O-Sequence.
---------------------------------------------------------

CHECKOSequence([1,3,5,7,9,11,14]);

[1, 3, 5, 7, 9, 11, 14]==> No, this is NOT an O-Sequence.
---------------------------------------------------------
```

Hence, the first sequence $(1, 4, 7, 10, 13, 6, 3)$ is an O-sequence, but the second sequence $(1, 4, 7, 10, 13, 17)$ is not an O-sequence.

Here we need the following algorithm, which we can obtain the differentiable O-sequence from an O-sequence made by Algorithm 2.1.

Algorithm 2.3 (CoCoA: Adding Up O-sequences)**.**

```
Define ADDUPHilbert(L)
   S:=[];
    For I := 2 To Len(L)
      Do S1:=Sum(First(L,I));
      Append(S,S1);
    EndFor;
   S2:=Comp(S,Len(S));
   S:=[S];
   Append(S,S2);
   S;
EndDefine;
```

Example 2.4 (CoCoA)**.** If we also run `ADDUPHilbert(L)` from CoCoA, then we can add up if a given O-sequence as follows:

```
ADDUPHilbert([1,3,5,7,9,11,4,2]);
[[4, 9, 16, 25, 36, 40, 42], 42]
---------------------------------------
```

So, we can obtain the differentiable O-sequence $(1, 4, 9, 16, 25, 36, 40, 42)$.

Theorem 2.5 (Theorem 4.8A, [10])**.** *Let $h = (h_0, h_1, \ldots, h_e)$ be the h-vector of a level algebra $A = R/\mathrm{Ann}(M)$ where $R = k[x_1, \ldots, x_r]$. Then, if F is a generic form of degree e, the level algebra $A = R/\mathrm{Ann}(\langle M, F\rangle)$ has h-vector $\mathbf{H} = (H_0, H_1, \ldots, H_e)$ where, for $i = 1, \ldots, e$,*

$$H_i = \min\left\{h_i + \binom{r-1+e-i}{e-i}, \binom{r-1+i}{i}\right\}.$$

Before we make an example, we introduce the following algorithm which we can obtain a level O-sequence using CoCoA based on Theorem 2.5.

Algorithm 2.6 (CoCoA: Obtaining Level h-vector)**.**

```
Define LEVELHVECTOR(T)
  NewT:=[1];
  R:=Comp(T,2);
  E:=Len(T)-1;
  For J := 2 To Len(T) Do
   I:=J-1;
          Ti:=Comp(T,J);
   T1:=Bin(R-1+E-I,E-I);
   T2:=Bin(R-1+I,I);
     NewTi:=Min(Ti+T1,T2);
    Append(NewT,NewTi);
  EndFor;
  Print "From h=", T, "   and r=", R, NewLine;
```

```
  Print "We have T=", NewT
EndDefine;
```

Example 2.7 (CoCoA)**.** Consider a level h-vector $(1, 4, 9, 16, 25, 36, 40, 42)$ in Example 2.4. Applying Algorithm 2.6, we have another level O-sequence as follows:

```
LEVELHVECTOR([1, 4, 9, 16, 25, 36, 40, 42]);
From h=[1, 4, 9, 16, 25, 36, 40, 42]   and r=4
We have T=[1, 4, 10, 20, 35, 46, 44, 43]
------------------------------------------------
```

So, we can obtain another level O-sequence $(1, 4, 10, 20, 35, 46, 44, 43)$.

3. The behavior of Local minima of a Non-Unimodal Level Artinian O-sequence of Codimension 3

Using the above Theorem 2.5, Kim showed how to produce a non-unimodal level O-sequence having the same local maxima as many as we want (see [9]).

Theorem 3.1 (Theorem 3.2, [9]). *Let* $\mathbf{H}'$ *be a Hilbert function such that*

$$\begin{aligned}\mathbf{\Delta H}' = \quad & (1,2,3,...,s+1, \\ & \underbrace{3^{N+1},...,3^{N+1}}_{3^{N+1}\text{-times}},\underbrace{3^{N},...,3^{N}}_{3^{N}\text{-times}}, \\ & \vdots \\ & \underbrace{3^{3},...,3^{3}}_{3^{3}\text{-times}},\underbrace{3^{2},...,3^{2}}_{3^{2}\text{-times}},\underbrace{3,3,3}_{3\text{-times}})\end{aligned}$$

where $s \gg 0$ *and* $\ell = 0,1,\cdots,N-1$, *and let* $\mathbf{H}$ *be in Theorem 2.5. Then* $\mathbf{H}$ *is a level O-sequence of codimension* 3 *having the same N-local maxima.*

Moreover, she proved that the Hilbert function in Theorem 3.1 is symmetric between any two local maxima.

Theorem 3.2 (Theorem 3.5, [9])**.** *Let* **H** *be as in Theorem 3.1. Then the Hilbert function* **H** *is symmetric between any two local maxima.*

From the previous theorem, we have a natural question here.

Question 3.3. Do any local minima of the Hilbert function **H** exist in two consecutive degrees?

We have a positive answer to Question 3.3 in Theorem 3.4.

Theorem 3.4. *Let* **H** *be as in Question 3.3. Then the Hilbert function* **H** *has local minima in two consecutive degrees.*

Proof. Let $\beta_\ell = s + 3^{N+1} + \cdots + 3^{N-\ell+1}$, $\beta_{\ell+1} = s + 3^{N+1} + \cdots + 3^{N-\ell}$.

Note that, by Theorem 3.1, $H_{\beta_\ell}, H_{\beta_{\ell+1}}$ are local maxima of the Hilbert function **H**.

Let $\alpha_\ell = \beta_\ell + \frac{3^{N-\ell}-1}{2}$.

First, we shall show that

$$H_{\alpha_\ell} = H_{\alpha_\ell+1}.$$

Let $k = \frac{3^{N-\ell}-1}{2}$. Then, by Theorem 2.5

$$\begin{aligned} H_{\alpha_\ell} &= \binom{s+2}{2} + \underbrace{3^{N+1} + \cdots + 3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &+ \underbrace{3^{N-\ell+1} + \cdots + 3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{k\text{-times}} \\ &+ \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - k}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - k}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} H_{\alpha_\ell+1} &= \binom{s+2}{2} + \underbrace{3^{N+1} + \cdots + 3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &+ \underbrace{3^{N-\ell+1} + \cdots + 3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{k+1\text{-times}} \\ &+ \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - (k+1)}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - (k+1)}. \end{aligned}$$

Hence we have

$$\begin{aligned} H_{\alpha_\ell+1} - H_{\alpha_\ell} &= \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{k+1\text{-times}} + \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - (k+1)}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - (k+1)} \\ &- \underbrace{(3^{N-\ell} + \cdots + 3^{N-\ell})}_{k\text{-times}} - \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - k}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - k} \end{aligned}$$

$$
\begin{aligned}
&= 3^{N-\ell} \\
&\quad +\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell+1}-(k+1)\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell+1}-(k+1)\right) \\
&\quad -\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell+1}-k\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell+1}-k\right) \\
&= 3^{N-\ell}+\left(-\frac{1}{8}+\frac{3^{2N-2\ell+2}}{8}-\frac{3^{N-\ell+1}}{2}(k+1)+\frac{(k+1)^2}{2}\right) \\
&\quad -\left(-\frac{1}{8}+\frac{3^{2N-2\ell+2}}{8}+\frac{3^{N-\ell+1}}{2}k+\frac{k^2}{2}\right) \\
&= 3^{N-\ell}-\frac{3^{N-\ell+1}}{2}+k+\frac{1}{2} \\
&= 3^{N-\ell}\left(1-\frac{3}{2}\right)+\frac{3^{N-\ell}-1}{2}+\frac{1}{2} \\
&= 0,
\end{aligned}
$$

which means that

$$H_{\alpha_\ell} = H_{\alpha_\ell+1},$$

as we wanted.

Now we shall show that H_{α_ℓ} and $H_{\alpha_\ell+1}$ are local minima.

In other words, it suffice to show

$$H_{\alpha_\ell-1} > H_{\alpha_\ell} = H_{\alpha_\ell+1} < H_{\alpha_\ell+2}.$$

Since $H_{\alpha_\ell+2} = H_{\alpha_\ell-1}$ by Theorem 3.2, it is enough to show that

$$H_{\alpha_\ell-1} > H_{\alpha_\ell}.$$

In fact, by Theorem 2.5,

$$\begin{aligned} H_{\alpha_\ell-1} &= \binom{s+2}{2} + \underbrace{3^{N+1} + \cdots + 3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &\quad + \underbrace{3^{N-\ell+1} + \cdots + 3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{(k-1)\text{-times}} \\ &\quad + \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - (k-1)}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - (k-1)}, \end{aligned}$$

and so we have

$$
\begin{aligned}
H_{\alpha_\ell - 1} - H_{\alpha_\ell} &= \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{(k-1)\text{-times}} + \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - (k-1)}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - (k-1)} \\
&\quad - \underbrace{\left(3^{N-\ell} + \cdots + 3^{N-\ell}\right)}_{k\text{-times}} - \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - k}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - k} \\
&= -3^{N-\ell} \\
&\quad + \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{2}\cdot 3^{N-\ell+1} - (k-1)\right)\left(\frac{1}{2} + \frac{1}{2}\cdot 3^{N-\ell+1} - (k-1)\right) \\
&\quad - \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{2}\cdot 3^{N-\ell+1} - k\right)\left(\frac{1}{2} + \frac{1}{2}\cdot 3^{N-\ell+1} - k\right) \\
&= -3^{N-\ell} + \left(-\frac{1}{8} + \frac{3^{2N-2\ell+2}}{8} - \frac{3^{N-\ell+1}}{2}(k-1) + \frac{(k-1)^2}{2}\right) \\
&\quad - \left(-\frac{1}{8} + \frac{3^{2N-2\ell+2}}{8} - \frac{3^{N-\ell+1}}{2}k + \frac{k^2}{2}\right) \\
&= -3^{N-\ell} + \frac{3^{N-\ell+1}}{2} - k + \frac{1}{2} \\
&= 3^{N-\ell}\left(-1 + \frac{3}{2}\right) - \frac{3^{N-\ell} - 1}{2} + \frac{1}{2} \\
&= \frac{3^{N-\ell}}{2} - \frac{3^{N-\ell} - 1}{2} + \frac{1}{2} \\
&= \frac{3^{N-\ell}}{2} - \frac{3^{N-\ell}}{2} + \frac{1}{2} + \frac{1}{2} \\
&= 1 \\
&> 0,
\end{aligned}
$$

which follows that $H_{\alpha_\ell - 1} > H_{\alpha_\ell}$.

Therefore, we have that

$$H_{\alpha_\ell - 1} > H_{\alpha_\ell} = H_{\alpha_\ell + 1} < H_{\alpha_\ell + 2},$$

as we wished. □

Now we are ready to prove the main theorem here.

Theorem 3.5. *Let* $\mathbf{H}$ *and* H_{α_ℓ} *be as in Theorem 3.4. Then all N-local minima are increasing.*

In other words,

$$H_{\alpha_\ell} < H_{\alpha_{\ell+1}}.$$

Proof. Let β_ℓ and α_ℓ be as above, and let $k_1 = \frac{3^{N-\ell}-1}{2}$, $k_2 = \frac{3^{N-\ell-1}-1}{2}$.

Then,

$$\begin{aligned} H_{\alpha_\ell} &= \binom{s+2}{2} + \underbrace{3^{N+1} + \cdots + 3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &\quad + \underbrace{3^{N-\ell+1} + \cdots + 3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{k_1\text{-times}} \\ &\quad + \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - k_1}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - k_1}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} H_{\alpha_{\ell+1}} &= \binom{s+2}{2} + \underbrace{3^{N+1} + \cdots + 3^{N+1}}_{3^{N+1}\text{-times}} + \cdots \\ &\quad + \underbrace{3^{N-\ell+1} + \cdots + 3^{N-\ell+1}}_{3^{N-\ell+1}\text{-times}} + \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{3^{N-\ell}\text{-times}} \\ &\quad + \underbrace{3^{N-\ell-1} + \cdots + 3^{N-\ell-1}}_{k_2\text{-times}} + \binom{\frac{3^{N-\ell}}{2} + \frac{1}{2} - k_2}{\frac{3^{N-\ell}}{2} - \frac{3}{2} - k_2}, \end{aligned}$$

and so

$$\begin{aligned} H_{\alpha_{\ell+1}} - H_{\alpha_\ell} &= \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{3^{N-\ell}\text{-times}} + \underbrace{3^{N-\ell-1} + \cdots + 3^{N-\ell-1}}_{k_2\text{-times}} \\ &\quad + \binom{\frac{3^{N-\ell}}{2} + \frac{1}{2} - k_2}{\frac{3^{N-\ell}}{2} - \frac{3}{2} - k_2} - \underbrace{3^{N-\ell} + \cdots + 3^{N-\ell}}_{k_1\text{-times}} \\ &\quad - \binom{\frac{3^{N-\ell+1}}{2} + \frac{1}{2} - k_1}{\frac{3^{N-\ell+1}}{2} - \frac{3}{2} - k_1} \end{aligned}$$

$$
\begin{aligned}
&= \left(3^{N-\ell}-k_1\right)3^{N-\ell}+3^{N-\ell-1}\cdot k_2\\
&\quad +\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell}-k_2\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell}-k_2\right)\\
&\quad -\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell+1}-k_1\right)\left(\frac{1}{2}+\frac{1}{2}\cdot 3^{N-\ell+1}-k_1\right)\\
&= \left(3^{N-\ell}-k_1\right)3^{N-\ell}+3^{N-\ell-1}\cdot k_2\\
&\quad +\left(-\frac{1}{8}+\frac{3^{2N-2\ell}}{8}-\frac{3^{N-\ell}}{2}k_2+\frac{{k_2}^2}{2}\right)\\
&\quad -\left(-\frac{1}{8}+\frac{3^{2N-2\ell+2}}{8}-\frac{3^{N-\ell+1}}{2}k_1+\frac{{k_1}^2}{2}\right)\\
&= 3^{2N-2\ell}-3^{N-\ell}k_1+3^{N-\ell-1}k_2\\
&\quad +\frac{3^{2N-2\ell}}{8}(1-3^2)-\frac{3^{N-\ell}}{2}k_2+\frac{3^{N-\ell+1}}{2}k_1+\frac{1}{2}k_2^2-\frac{1}{2}k_1^2\\
&= 3^{N-\ell}\left(\frac{3}{2}k_1-k_1\right)+3^{N-\ell-1}\left(k_2-\frac{3}{2}k_2\right)+\frac{1}{2}k_2^2-\frac{1}{2}k_1^2\\
&= \frac{3^{N-\ell}}{2}k_1-\frac{3^{N-\ell-1}}{2}k_2++\frac{1}{2}k_2^2-\frac{1}{2}k_1^2\\
&= \frac{1}{2}k_1(3^{N-\ell}-k_1)+\frac{1}{2}k_2(k_2-3^{N-\ell-1})\\
&= \frac{1}{2}\cdot\frac{3^{N-\ell}-1}{2}\left(3^{N-\ell}-\frac{3^{N-\ell}-1}{2}\right)\\
&\quad +\frac{1}{2}\cdot\frac{3^{N-\ell-1}-1}{2}\left(\frac{3^{N-\ell-1}-1}{2}-3^{N-\ell-1}\right)\\
&= \frac{3^{N-\ell}-1}{4}\cdot\frac{3^{N-\ell}+1}{2}+\frac{3^{N-\ell-1}-1}{4}\cdot\frac{-3^{N-\ell}-1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{8}\cdot(3^{N-\ell}-1)(3^{N-\ell}+1) \\
&\quad +\frac{1}{8}\cdot(3^{N-\ell-1}-1)(-3^{N-\ell-1}+1) \\
&= \frac{1}{8}\cdot(3^{2N-2\ell}-1)+\frac{1}{8}\cdot(1-3^{2N-2\ell-2}) \\
&= \frac{1}{8}\cdot 3^{2N-2\ell}-\frac{1}{8}\cdot 3^{2N-2\ell-2} \\
&= \frac{1}{8}\cdot 3^{2N-2\ell}(1-3^{-2}) \\
&= 3^{2(N-\ell-1)} \\
&> 0,
\end{aligned}
$$

which means that

$$H_{\alpha_\ell} < H_{\alpha_{\ell+1}}$$

as we desired. □

Example 3.6 (CoCoA)**.** Let $\mathbf{H}'$ be an O-sequence of codimension 3 such that

$$\mathbf{\Delta H}' = (1,2,3,...,200,\underbrace{81,\cdots,81}_{\text{81-times}},\underbrace{27,\cdots,27}_{\text{27-times}},\underbrace{9,\cdots,9}_{\text{9-times}},\underbrace{3,3,3}_{\text{3-times}}).$$

Then using Thoerem 2.5 and Algorithm 2.6, we obtain another level O-sequence as follows:

$$
\begin{aligned}
&\mathbf{H} \\
= \ &(1, 3, 6, \cdots, 27402, 27441, \\
&\mathbf{27481}, 27468, 27456, 27445, 27435, 27426, 27418, 27411, 27405, \\
&27400, 27396, 27393, 27391, \mathbf{27390, 27390}, 27391, 27393, 27396, \\
&27400, 27405, 27411, 27418, 27426, 27435, 27445, 27456, 27468, \\
&\mathbf{27481}, 27477, 27474, 27472, \mathbf{27471, 27471}, 27472, 27474, 27477, \\
&\mathbf{27481}, \mathbf{27480, 27480}, \mathbf{27481}),
\end{aligned}
$$

which shows that the local minima 27390, 27471 and 27480 are increasing as we proved in Theorem 3.5. Moreover, the difference between any two local minima are

$$
\begin{aligned}
&27471 - 27390 = 3^4, \quad \text{and} \\
&27480 - 27471 = 3^2,
\end{aligned}
$$

as we have see in the proof of Theorem 3.5.

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ABSTRACT

The Behavior of Local Minima of A Level O-sequence of Codimension 3 Having The Same Local Maxima

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F. Zanello found that how to make a non-unimodal level O-sequence of codimension 3 in [12]

$$(1, 3, \dots t,\ t, t+1, t, t, t+1, \dots t, t, t+1)$$

having local maxima as many as we want.

In [9], using a different method from Zanello's construction, Kim found a new construction of a non-unimodal level O-sequence of codimension 3 having the same N-local maxima at any time. In

addition, the level O-sequence she found is symmetric between any two local maxima.

Using a new construction of a non-unimodal level O-sequence which Kim produced in [9], we prove any local minima of the level O-sequence exist in two consecutive degrees. In addition, we prove local minima of a level O-sequence are increasing.

A computer program CoCoA was used for all example in this thesis.