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석사학위 청구논문

Some Example of a Non-level Artinian O-sequence ending 6 4 3

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교육학과 수학교육전공
이혜인

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이 논문을 석사학위 논문으로 제출함.

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인준서

이혜인의 석사학위 논문으로 인준함.

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논문개요

참고논문 7번에서 A.V Geramita, T.Harima, J.C. Migliore 그리고 Y.S. Shin은 여차원 3인 O-수열이

1, 3, ..., ,6, 3, 2

또는

1, 3, ... ,5, 3, 2

형태이면 Level O-수열이 되지 않는다는 것을 증명하였다.

본 논문에서는 Inverse System (또는 Macaulay Duality)를 이용하여 여차원 3인 O-수열

1, 3, 4, 5, ... ,6, 4, 3

가 Level이 되지 않는다는 것을 증명하였다.

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ABSTRACT

Some Example of a Non-level Artinian O-sequence ending 6 4 3

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A.V Geramita, T.Harima, J.C. Migliore and Y.S. Shin proved that if a level O-sequnce of codimension 3 of the form

1, 3, ..., ,6, 3, 2

or

1, 3, ... ,5, 3, 2

are not level in [7].

in this paper, we proved a level O-sequnce of codimention 3

1, 3, 4, 5, ... ,6, 4, 3

is not level using Inverse System or Macaulay duality.

1. Introduction

Let $R = k[x_0, x_1, \cdots, x_n] = \oplus_{i\geq 0} R_i$, k an algebraically closed field of characteristic 0, and let I be a homogeneous ideal of R, $A = R/I$. The ***Hilbert function*** *of* A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R. The ***h-vector*** of A is $h(A) := (h_0, h_1, \ldots, h_s)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and s is the last index such that $\dim_k A_k \neq 0$. We call s the ***socle degree*** A. Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the ***codimension*** of A.

Let $R = k[x_0, \ldots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. A is a ***level algebra*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$.

The sequence $\{h_i\}_{i\geq 0}$ (with $h_0 = 1$ and $h_1 \leq n$) is called an ***O-sequence*** if there is a homogeneous ideal $I \subset R$ such that if $A = R/I$ then $\mathbf{H}_A(i) = h_i$. In particular, the h-vector $h = (1, n, h_2, \cdots, h_s)$ is called an ***O-sequence*** if there is an Artinian quotient A of R whose h-vector is h. The h-vector $h = (1, n, h_2, \cdots, h_s)$ is called a ***level O-sequence*** if there is an Artinian level algebra having h-vector h (see

[1]–[5], [7]–[9], [11], [13], [14]). Moreover, we say that the sequence is a ***Gorenstein sequence*** if it is a level O-sequence with $h_s = 1$ (see [2], [4], [9]). When $h_s \neq 0$ we say that $s+1$ is the ***length*** of the sequence.

For graded Artinian level algebras, it has been recently studied (see [2]–[5], [7], [8], [11], [13], [14]).

Let A be a ring and M an A-module. An element a of A is said ***M-regular*** if it is not a zero-divisor on M, i.e., if $M \xrightarrow{a} M$ is injective. The set of the M-regular elements is a multiplicative subset of A.

We say $a_1, a_2, \cdots, a_r$ is an ***M-regular sequence*** (or simply M-sequence) if the following conditions are satisfied:

(1) $\forall 1 \leq i \leq r$, a_1 is not a zero-divisor on $M/(a_1, a_2, \cdots, a_{i-1})M$,

(2) $M \neq aM$.

When all a_i belong to an ideal I we say $a_1, a_2, \cdots, a_r$ is an ***M-regular sequence*** in I.

A total order on the monomials of each degree is said to be a ***term order*** if

(1) $x_1 > \cdots > x_n$, and

(2) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$ for any monomials m, m_1 and m_2.

The ***lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

(1) $\sum i_t > \sum j_t$ or

(2) $\sum i_t = \sum j_t$ and there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.

Let S be a subset of all monomials in R. S is a ***lex-segment*** if a monomial m of degree d is in S, then every monomial m' of degree d in R_d such that $m' > m$ is in S. Let $I = \bigoplus_{t\geq 0} I_t$ be a graded ideal of R. We say that I is a ***lex-segment ideal*** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

Let I be a lex-segment ideal of $R = k[x, y, z]$ with Hilbert function $\mathbf{H} = (1, 3, 4, 5, 6, 4, 3)$. Then the minimal free resolution of R/I is

$0 \rightarrow R^2(-7) \oplus R(-8) \oplus R^3(-9) \rightarrow R(-3) \oplus R^5(-6) \oplus R^2(-7) \oplus R^6(-8) \rightarrow R^2(-2) \oplus R^3(-5) \oplus R(-6) \oplus R^3(-7) \rightarrow R \rightarrow R/I \rightarrow 0.$

As we see from the minimal free resolution of R/I above, we cannot say if the Hilbert function $\mathbf{H}$ is level or not.

The goal of this thesis is to prove that the h-vector

$$(1, 3, 4, 5, \ldots, 6, 4, 3)$$

cannot be level in general (see Theorem 7) using the theory of ***inverse systems*** or ***Macaulauy duality*** (see [6] for more details).

2. Inverse Systems

We now recall a very interesting method for construsting Artinian level algebras. This method is based on the idea of ***Macaulay's Inverse Systems***. We will only give a quick review of the method and refer the reader [6] for more details.

Let $R = k[x_1, \cdots, x_n]$ and $S = k[y_1, \cdots, y_n]$. We can consider S as a graded R-module by: if $F \in S_j$ then $x_i \circ F = (\frac{\partial}{\partial y_i})F$. We extend this action in the obvious way and note that the action *lowers* degree on S and hence S is not a finitely generated R-module.

There is an order reversing function from the ideals of R to the R-submodules of S defined by:

$$\varphi_1 : \{\text{ideals of } R\} \rightarrow \{R\text{-submodules of } S\}$$

where

$$\varphi_1(I) = \{F \in S | G \circ F = 0 \text{ for all } G \in I\}$$

This is a 1-1 correspondence whose inverse (φ_2) is given by $\varphi_2(M) = \operatorname{ann}_R(M) = \{r \in R \mid r \cdot x = 0, \forall x \in M\}$. In fact, we denote $\varphi_1(I)$ by I^{-1}, which is called the ***inverse system*** to I.

It is very easy to construct I^{-1} (and this is at the heart of the proof of the 1-1 correspondence). One first observes that the pairing

$$R_j \times S_j \longrightarrow S_0 \simeq k$$

is a perfect pairing and so S_j can be identified with R_j^* (the dual vector space to R_j). If V is a subspace of R_j we write $V^\perp$ for the annihilator of V in this pairing. Then, if $I \subset R$ is an ideal and I_j its j^{th} graded piece, then Macaulay observed that:

$$(I^{-1})_j = I_J^\perp.$$

It follows immediately that

$$\dim_k (I^{-1})_j = \dim_k R_j - \dim_k I_j = \mathbf{H}(R/I, j).$$

It is a simple consequence of this last observation that I^{-1} is a finitely generated R-submodule of S if and only if R/I is Artinian.

Remark 2.1. There is another way to define Inverse Systems which considers S as an R-module in a different way. In this other method, we consider the ***contraction*** operations, D_{x_i} where, if F is a monomial in S_j then

$$D_{x_i}(F) = \begin{cases} 0, & \text{if } y_i \text{ does not divide F,} \\ F/y_i & \text{if } y_i \text{ divides F.} \end{cases}$$

We extend this action to all of S in the obvious way and recall that when the characteristic of k is 0, this action is equivalent to the one described above. The contraction operation has the advantage that it doesn't end up increasing the sizes of coefficients (see [6] for more details).

The really interesting connection between inverse systems and what we've been considering is the following theorem of Macaulay. We continue with notations as above.

Theorem 2.2 (Macaulay)**.** *Let I be an Artinian ideal of R and I^{-1} its inverse system. Then I^{-1} has exactly ν_j minimal generators of degree j if and only if the socle of R/I in degree j has dimension exactly ν_j.*

Remark 2.3. 1) This gives us a new interpretation of the socle vector of an Artinian algebra of the form $A = k[x_1, \cdots, x_n]/I$. The entries of the socle vector tell us the number of generators of the inverse system of I in each degree.

2) Since we are interested in level algebras (Atrinian, say, with socle degree s, type c and embedding dimension n) then this theorem tells us how to make ***all*** of them. We look at every subspace of $S_s = k[y_1, \cdots, y_n]_s$ of dimension c and form the R-submodule of S generated by that subspace. The result is a level algebra of the type we are looking for and every level algebra of socle degree s, type c and embedding dimension n arises in this way.

Example 2.4. Suppose we would like to construct a level algebra with socle degree 4, embedding dimension 3 and type 2. Macaulay's Theorem says we have to look at a two dimensional vector space of S_4, where $S = k[y_1, y_2, y_3]$ and take the inverse system it generates.

For example, consider the vector space of S_4 generated by $F_1 = y_1^4$, $F_2 = y_2^4 + y_3^4$ and $F_3 = y_1 y_2 y_3^2$. The inverse system, call it M generated by these three elements of degree 4 will have $M_3 = \langle y_1^3, y_2^3, y_3^3, y_1 y_3^2, y_2 y_3^2, y_1 y_2 y_3 \rangle$, $M_2 = \langle y_1^2, y_2^2, y_3^2, y_1 y_2, y_2 y_3, y_1 y_3 \rangle$, $M_1 = \langle y_1, y_2, y_3 \rangle$, and $M_0 = \langle 1 \rangle$. So, if $I = \text{ann}_R(M)$ and $A = k[x_1, x_2, x_3]/I$ then the h-vector of A is $(1, 3, 6, 6, 3)$.

3. NON EXISTENCE OF A LEVEL O-SEQUENCE

First, we need the following lemma to prove the main theorem.

Lemma 3.1. *Let* $R = k[x, y, z]$*. If* I *is a homogeneous ideal of* R *such that* R/I *has the Hilbert function* $\mathbf{H} = (1, 3, 4, 5, \dots)$*, then two minimal generators of* I *in degree 2 have a linear common factor.*

Proof. Let I be an ideal in $R = k[x, y, z]$ so that the Hilbert function of $A = R/I$ is

$$1 \quad 3 \quad 4 \quad 5 \quad \cdots$$

Let $\langle F,\, G\rangle = I_2$ and assume that F and G be a regular sequence. Since 4 and 5 have a maximal growth in degrees 2 and 3, we see that I does not have any generators in degree 3, that is,

$$\begin{aligned} R_1 I_2 &= I_3 \\ &= \langle Fx, Fy, Fz, Gx, Gy, Gz\rangle. \end{aligned}$$

Since $\dim_k I_3 = 5 = 10 - \mathbf{H}(R/I, 3)$, we have that one of Fx, Fy, Fz, Gx, Gy and Gz is a linear combination of the rest of 5 elements.

Without loss of generality, we may assume that

$$Fx \in \langle Fy, Fz, Gx, Gy, Gz\rangle.$$

Then

$$Fx = aFy + bFz + cGx + dGy + eGz$$

where a, b, c, d, $e \in k$, and so

$$F \cdot (x - ay - bz) = G \cdot (cx + dy + ez) \in (G)$$

Since F and G are a regular sequence, we have that

$$(x - ay - bz) \in (G),$$

which is a contradiction since $\deg(G) = 2$. In other words, F and G cannot be a regular sequence.

Note that $k[x, y, z]$ is a unique factorization domain. Hence

$$(x - ay - bz) \mid F \cdot (x - ay - bz) = G \cdot (cx + dy + ez)$$

implies

$$(x - ay - bz) \mid G \quad \text{or} \quad (x - ay - bz) \mid (cx + dy + ez).$$

However, if $(x - ay - bz) \mid (cx + dy + ez)$, then $F = \alpha \cdot G$ for some $\alpha \in k - (0)$, that is, $\dim_k \langle F, G \rangle = 1$, a contradiction.

Thus $(x - ay - bz) \mid G$, that is, $G = (x - ay - bz) \cdot L$ for some linear form $L \in R_1$ and so

$$\begin{aligned} & F \cdot (x - ay - bz) = G \cdot (cx + dy + ez) = (cx + dy + ez) \cdot L \cdot (x - ay - bz) \\ \Rightarrow \quad & F = (cx + dy + ez) \cdot L. \end{aligned}$$

In other words, F and G have a linear common factor $L \in R_1$, as we wished. $\square$

Now we are ready to prove the main result.

Theorem 3.2. *The h-vector* $\mathbf{H} = (1, 3, 4, 5, \ldots 6, 4, 3)$ *is not a level h-vector.*

Proof. First we shall prove that $\mathbf{H} = (1, 3, 4, 5, 6, 4, 3)$ cannot be level.

If it were, let I be an Artinian level ideal in $R = k[x, y, z]$ so that the Hilbert function of $A = R/I$ is

$$1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 4 \quad 3 \quad 0 \quad \rightarrow.$$

By Lemma 5, two forms in I_2 have to have a linear common factor. Without loss of generality, we may assume that

$$I_2 = \langle xy, xz \rangle \quad \text{or} \quad \langle x^2, xy \rangle.$$

Then $R_1 I_2 = \langle x^2y, x^2z, xy^2, xyz, xz^2 \rangle$ or $\langle x^3, x^2y, xy^2, x^2z, xyz \rangle$. In other words,

$$\mathbf{H}(R/R_1I_2, 3) = \dim_k R_3 - \dim_k R_1I_2 = 10 - 5 = 5 = \mathbf{H}(R/I, 3).$$

This means that $I_3 = R_1I_2$. Similarly, we have that

$$\begin{aligned} R_1I_3 = \quad & \langle x^3y, x^3z, x^2y^2, x^2yz, x^2z^2, xy^3, xy^2z, xyz^2, xz^3 \rangle \\ \text{or} \quad & \\ & \langle x^4, x^3y, x^2y^2, x^3z, x^2yz, xy^3, xy^2z, x^2z^2, xyz^2 \rangle. \end{aligned}$$

Thus

$$\mathbf{H}(R/R_1I_3, 4) = \dim_k R_4 - \dim_k(R_1I_3) = 15 - 9 = 6 = \mathbf{H}(R/I, 4),$$

i.e., $I_4 = R_1I_3$.

Hence, I has no generators in degrees 3 and 4. I.e., we have that $(I_{\leq 4}) = (I_{\leq 2}) = (xy, xz)$ or (x^2, xy).

Now we suppose $(I_{\leq 4}) = (xy, xz)$.

Let $S = k[X, Y, Z]$ and F, G,$H \in S_6$ be such that $\langle F, G, H \rangle^{\perp} = I$. Since $(I_{\leq 4}) = (xy, xz)$, we may assume that

$$F, G, H \in \langle Y^6, Y^5Z, Y^4Z^2, Y^3Z^3, Y^2Z^4, YZ^5, Z^6, X^6\rangle.$$

So, let

$$F = a_1Y^6 + a_2Y^5Z + a_3Y^4Z^2 + a_4Y^3Z^3 + a_5Y^2Z^4 + a_6YZ^5 + a_7Z^6$$
$$+aX^6,$$

$$\begin{aligned} G &= b_1Y^6 + b_2Y^5Z + b_3Y^4Z^2 + b_4Y^3Z^3 + b_5Y^2Z^4 + b_6YZ^5 + b_7Z^6, \\ H &= c_1Y^6 + c_2Y^5Z + c_3Y^4Z^2 + c_4Y^3Z^3 + c_5Y^2Z^4 + c_6YZ^5 + c_7Z^6, \end{aligned}$$

where $a_i, b_j, c_\ell \in k$ for every i, j and ℓ. If F, G and H are linear combinations in $k[Y, Z]_6$, then the fifth contractions of F, G and H have at most two variables, that is, Y and Z. This means that $\mathbf{H}(A, 1) \leq 2$, which is a contradiction since $\mathbf{H}(A, 1) = 3$. Hence we cannot have three forms F,G and H in $k[Y, Z]_6$. Thus, without loss of generality, we may assume $a = 1$.

Now consider all the contractions of F, G and H. They may be viewed as vectors in k^7 as follows:

$$\begin{aligned} D_X(F) &= X^5 \\ &\leftrightarrow (0, 0, 0, 0, 0, 0, 1) \end{aligned}$$

$$\begin{aligned} D_Y(F) &= a_1Y^5 + a_2Y^4Z + a_3Y^3Z^2 + a_4Y^2Z^3a_5YZ^4 + a_6Z^5 \\ &\leftrightarrow (a_1, a_2, a_3, a_4, a_5, a_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(F) &= a_2Y^5 + a_3Y^4Z + a_4Y^3Z^2 + a_5Y^2Z^3a_6YZ^4 + a_7Z^5 \\ &\leftrightarrow (a_2, a_3, a_4, a_5, a_6, a_7, 0) \end{aligned}$$

$$\begin{aligned} D_X(G) &= 0 \\ &\leftrightarrow (0,0,0,0,0,0,0) \end{aligned}$$

$$\begin{aligned} D_Y(G) &= b_1Y^5 + b_2Y^4Z + b_3Y^3Z^2 + b_4Y^2Z^3 + b_5YZ^4 + b_6Z^5 \\ &\leftrightarrow (b_1, b_2, b_3, b_4, b_5, b_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(G) &= b_2Y^5 + b_3Y^4Z + b_4Y^3Z^2 + b_5Y^2Z^3 + b_6YZ^4 + b_7Z^5 \\ &\leftrightarrow (b_2, b_3, b_4, b_5, b_6, b_7, 0). \end{aligned}$$

$$\begin{aligned} D_X(H) &= 0 \\ &\leftrightarrow (0,0,0,0,0,0,0) \end{aligned}$$

$$\begin{aligned} D_Y(H) &= c_1Y^5 + c_2Y^4Z + c_3Y^3Z^2 + c_4Y^2Z^3 + c_5YZ^4 + c_6Z^5 \\ &\leftrightarrow (c_1, c_2, c_3, c_4, c_5, c_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(H) &= c_2Y^5 + c_3Y^4Z + c_4Y^3Z^2 + c_5Y^2Z^3 + c_6YZ^4 + c_7Z^5 \\ &\leftrightarrow (c_2, c_3, c_4, c_5, c_6, c_7, 0). \end{aligned}$$

Since $\mathbf{H}(R/I, 5) = 4$, we have that a matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & 0 \\ c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & 0 \end{pmatrix}$$

has rank 4, and hence

$$\mathbf{A_1} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \end{pmatrix}$$

has rank 3.

We now consider the double contractions of F and G, where we identify a polynomial

$$\alpha_1 Y^4 + \alpha_2 Y^3 Z + \alpha_3 Y^2 Z^2 + \alpha_4 Y Z^3 + \alpha_5 Z^4 + \beta X^4$$

as a vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta) \in k^6$. Then we obtain

$$\begin{array}{rcl} D_{X,X}(F) & = & X^4 \\ & \leftrightarrow & (0,0,0,0,0,1) \\ \\ D_{Y,Y}(F) & = & a_1 Y^4 + a_2 Y^3 Z + a_3 Y^2 Z^2 + a_4 Y Z^3 + a_5 Z^4 \\ & \leftrightarrow & (a_1, a_2, a_3, a_4, a_5, 0) \\ \\ D_{Z,Y}(F) & = & a_2 Y^4 + a_3 Y^3 Z + a_4 Y^2 Z^2 + a_5 Y Z^3 + a_6 Z^4 \\ & \leftrightarrow & (a_2, a_3, a_4, a_5, a_6, 0) \\ \\ D_{Z,Z}(F) & = & a_3 Y^4 + a_4 Y^3 Z + a_5 Y^2 Z^2 + a_6 Y Z^3 + a_7 Z^4 \\ & \leftrightarrow & (a_3, a_4, a_5, a_6, a_7, 0) \\ \\ D_{Y,Y}(G) & = & b_1 Y^4 + b_2 Y^3 Z + b_3 Y^2 Z^2 + b_4 Y Z^3 + b_5 Z^4 \\ & \leftrightarrow & (b_1, b_2, b_3, b_4, b_5, 0) \\ \\ D_{Z,Y}(G) & = & b_2 Y^4 + b_3 Y^3 Z + b_4 Y^2 Z^2 + b_5 Y Z^3 + b_6 Z^4 \\ & \leftrightarrow & (b_2, b_3, b_4, b_5, b_6, 0) \\ \\ D_{Z,Z}(G) & = & b_3 Y^4 + b_4 Y^3 Z + b_5 Y^2 Z^2 + b_6 Y Z^3 + b_7 Z^4 \\ & \leftrightarrow & (b_3, b_4, b_5, b_6, b_7, 0) \end{array}$$

$$\begin{array}{rcl} D_{Y,Y}(H) & = & c_1Y^4 + c_2Y^3Z + c_3Y^2Z^2 + c_4YZ^3 + c_5Z^4 \\ & \leftrightarrow & (c_1, c_2, c_3, c_4, c_5, 0) \end{array}$$

$$\begin{array}{rcl} D_{Z,Y}(H) & = & c_2Y^4 + c_3Y^3Z + c_4Y^2Z^2 + c_5YZ^3 + c_6Z^4 \\ & \leftrightarrow & (c_2, c_3, c_4, c_5, c_6, 0) \end{array}$$

$$\begin{array}{rcl} D_{Z,Z}(H) & = & c_3Y^4 + c_4Y^3Z + c_5Y^2Z^2 + c_6YZ^3 + c_7Z^4 \\ & \leftrightarrow & (c_3, c_4, c_5, c_6, c_7, 0). \end{array}$$

Since $\mathbf{H}(R/I, 4) = 6$, we see that a matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_3 & b_4 & b_5 & b_6 & b_7 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \\ c_2 & c_3 & c_4 & c_5 & c_6 & 0 \\ c_3 & c_4 & c_5 & c_6 & c_7 & 0 \end{pmatrix}$$

has rank 6, and so

$$\mathbf{B_1} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ b_2 & b_3 & b_4 & b_5 & b_6 \\ b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ c_2 & c_3 & c_4 & c_5 & c_6 \\ c_3 & c_4 & c_5 & c_6 & c_7 \end{pmatrix}$$

has rank 5.

Since the rank of $\mathbf{A}_1$ is 3, one of four row vectors of $\mathbf{A}_1$ has to be a linear combination of the other three.

Case 1–1. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle.$$

Since

$$\begin{array}{rl} & (a_2, a_3, a_4, a_5, a_6, a_7), (c_1, c_2, c_3, c_4, c_5, c_6), (c_2, c_3, c_4, c_5, c_6, c_7) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle, \end{array}$$

we have

$$\begin{array}{rl} & (a_2, a_3, a_4, a_5, a_6), (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle \end{array}$$

and

$$\begin{array}{rl} & (a_3, a_4, a_5, a_6, a_7), (c_2, c_3, c_4, c_5, c_6), (c_3, c_4, c_5, c_6, c_7) \\ \in & \langle (a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle. \end{array}$$

Then we have

$$\begin{array}{rl} & (a_3, a_4, a_5, a_6, a_7), (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6), (c_3, c_4, c_5, c_6, c_7) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle, \end{array}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 4, a contradiction.

One can show that the following five cases are also impossible by the same way as **Case 1–1** (see **Case 1–2** $\sim$ **1–6**).

Case 1–2. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (c_1, c_2, c_3, c_4, c_5, c_6), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle.$$

Case 1–3. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6) \rangle.$$

Case 1–4. The row space of

$$\mathbf{A_1} = \langle (b_1, b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5, c_6), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle.$$

Case 1–5. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (a_2, a_3, a_4, a_5, a_6, a_7), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle.$$

Case 1–6. The row space of

$$\mathbf{A_1} = \langle (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle.$$

Case 1–7. The row space of
$$\mathbf{A_1} = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle.$$

Since

$$\begin{aligned} &(a_1, a_2, a_3, a_4, a_5, a_6), (c_1, c_2, c_3, c_4, c_5, c_6), (c_2, c_3, c_4, c_5, c_6, c_7) \\ &\in \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle, \end{aligned}$$

we have

$$\begin{aligned} &(a_1, a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6) \\ &\in \langle (a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle \end{aligned}$$

and

$$(a_2, a_3, a_4, a_5, a_6), (c_2, c_3, c_4, c_5, c_6), (c_3, c_4, c_5, c_6, c_7)$$
$$\in \langle (a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle.$$

Thus

$$(a_1, a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6), (c_3, c_4, c_5, c_6, c_7)$$
$$\in \langle (a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7) \rangle,$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 4, a contradiction.

One can show that the following five cases are also impossible by the same way as **Case 1–7** (see **Case 1–8** $\sim$ **1–12**).

Case 1–8. The row space of

$$\mathbf{A_1} = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (c_1, c_2, c_3, c_4, c_5, c_6), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle$$

Case 1–9. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (a_2, a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6, b_7) \rangle$$

Case 1–10. The row space of

$$\mathbf{A_1} = \langle (b_2, b_3, b_4, b_5, b_6, b_7), (c_1, c_2, c_3, c_4, c_5, c_6), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle$$

Case 1–11. The row space of

$$\mathbf{A_1} = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (a_2, a_3, a_4, a_5, a_6, a_7), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle$$

Case 1–12. The row space of

$$\mathbf{A_1} = \langle (b_1, b_2, b_3, b_4, b_5, b_6), (b_2, b_3, b_4, b_5, b_6, b_7), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle$$

Case 1–13. The row space of

$$\mathbf{A_1} = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle.$$

Since

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6, b_7), (c_2, c_3, c_4, c_5, c_6, c_7) \\ \in \ & \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle, \end{aligned}$$

we have

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6) \\ \in \ & \langle (a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4, c_5) \rangle, \end{aligned}$$

and

$$\begin{aligned} & (a_2, a_3, a_4, a_5, a_6), (b_3, b_4, b_5, b_6, b_7), (c_3, c_4, c_5, c_6, c_7) \\ \in \ & \langle (a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6) \rangle. \end{aligned}$$

Thus

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5), (a_2, a_3, a_4, a_5, a_6), (b_3, b_4, b_5, b_6, b_7), \\ & (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6), (c_3, c_4, c_5, c_6, c_7) \\ \in \ & \langle (a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5), \rangle, \end{aligned}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 4, a contradiction.

We also prove that the following two cases are impossible (see **Case 1–14** $\sim$ **1–15**).

Case 1–14. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6, b_7), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle.$$

Case 1–15. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle.$$

Case 1–16. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6, b_7), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle.$$

Since

$$\begin{array}{cl} & (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6, b_7), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle, \end{array}$$

we have

$$\begin{array}{cl} & (a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4, c_5) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6) \rangle, \end{array}$$

and

$$\begin{array}{rl} & (a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6) \\ \in & \langle (a_2, a_3, a_4, a_5, a_6), (b_3, b_4, b_5, b_6, b_7), (c_3, c_4, c_5, c_6, c_7) \rangle. \end{array}$$

Hence

$$\begin{array}{rl} & (a_2, a_3, a_4, a_5, a_6), (a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5), \\ & (b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6), (b_3, b_4, b_5, b_6, b_7), (c_3, c_4, c_5, c_6, c_7), \rangle, \end{array}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 4, a contradiction.

We know that the following two cases are also impossible (see **Case 1–17 $\sim$ 1–18**).

Case 1–17. The row space of

$$\mathbf{A_1} = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_1, b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle.$$

Case 1–18. The row space of

$$\mathbf{A_1} = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6, b_7), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle.$$

Case 1–19. The row space of

$$\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle.$$

Since

$$\begin{aligned}
& (a_2, a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6, b_7), (c_2, c_3, c_4, c_5, c_6, c_7) \\
\in\ & \langle (a_1, a_2, a_3, a_4, a_5, a_6), (b_1, b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5, c_6) \rangle,
\end{aligned}$$

we have

$$\begin{aligned}
& (a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6) \\
\in\ & \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4, c_5) \rangle,
\end{aligned}$$

and

$$\begin{aligned}
& (a_3, a_4, a_5, a_6, a_7), (b_3, b_4, b_5, b_6, b_7), (c_3, c_4, c_5, c_6, c_7) \\
\in\ & \langle (a_2, a_3, a_4, a_5, a_6), (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6) \rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
& (a_2, a_3, a_4, a_5, a_6), (a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6), \\
& (b_3, b_4, b_5, b_6, b_7), (c_2, c_3, c_4, c_5, c_6), (c_3, c_4, c_5, c_6, c_7) \\
\in\ & \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5) \rangle,
\end{aligned}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 4, a contradiction.

Note that the following case is also impossible as **Case 1–19**.

Case 1–20. The row space of

$$\mathbf{A_1} = \langle (a_2, a_3, a_4, a_5, a_6, a_7), (b_2, b_3, b_4, b_5, b_6, b_7), (c_2, c_3, c_4, c_5, c_6, c_7) \rangle.$$

By **Case 1–1∼1–20**, $\mathbf{B_1}$ cannot have rank 5. Hence the O-sequence $\mathbf{H} = (1, 3, 4, 5, 6, 4, 3)$ cannot be level.

Now consider the case of $(I_{\leq 4}) = (x^2, xy)$.

Let F, G,$H \in S_6$ be such that $\langle F, G, H\rangle^{\perp} = I$ as above. Since $(I_{\leq 4}) = (x^2, xy)$, we may assume that

$$F, G, H \in \langle Y^6, Y^5Z, Y^4Z^2, Y^3Z^3, Y^2Z^4, YZ^5, Z^6, XZ^5\rangle.$$

So, let

$F = a_1Y^6+a_2Y^5Z+a_3Y^4Z^2+a_4Y^3Z^3+a_5Y^2Z^4+a_6YZ^5+a_7Z^6+aXZ^5$,

$G = b_1Y^6 + b_2Y^5Z + b_3Y^4Z^2 + b_4Y^3Z^3 + b_5Y^2Z^4 + b_6YZ^5 + b_7Z^6$,

$H = c_1Y^6 + c_2Y^5Z + c_3Y^4Z^2 + c_4Y^3Z^3 + c_5Y^2Z^4 + c_6YZ^5 + c_7Z^6$.

Since $\mathbf{H}(A, 1) = 3$ we cannot have both F , G and H in $k[Y, Z]_6$, and so we may assume $a = 1$.

Now consider all the contractions of F , G and H. They may be viewed as vectors in k^7 as follows:

$$\begin{aligned} D_X(F) &= Z^5 \\ &\leftrightarrow (0, 0, 0, 0, 0, 1, 0) \end{aligned}$$

$$\begin{aligned} D_Y(F) &= a_1Y^5 + a_2Y^4Z + a_3Y^3Z^2 + a_4Y^2Z^3a_5YZ^4 + a_6Z^5 \\ &\leftrightarrow (a_1, a_2, a_3, a_4, a_5, a_6, 0) \end{aligned}$$

$$\begin{aligned} D_Z(F) &= a_2Y^5 + a_3Y^4Z + a_4Y^3Z^2 + a_5Y^2Z^3a_6YZ^4 + a_7Z^5 + XZ^4 \\ &\leftrightarrow (a_2, a_3, a_4, a_5, a_6, a_7, 1) \end{aligned}$$

$$\begin{aligned} D_X(G) &= 0 \\ &\leftrightarrow (0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_Y(G) &= b_1Y^5 + b_2Y^4Z + b_3Y^3Z^2 + b_4Y^2Z^3 + b_5YZ^4 + b_6Z^5 \\ &\leftrightarrow (b_1, b_2, b_3, b_4, b_5, b_6, 0) \end{aligned}$$

$$
\begin{aligned}
D_Z(G) &= b_2Y^5 + b_3Y^4Z + b_4Y^3Z^2 + b_5Y^2Z^3 + b_6YZ^4 + b_7Z^5 \\
&\leftrightarrow (b_2, b_3, b_4, b_5, b_6, b_7, 0)
\end{aligned}
$$

$$
\begin{aligned}
D_X(H) &= 0 \\
&\leftrightarrow (0, 0, 0, 0, 0, 0, 0)
\end{aligned}
$$

$$
\begin{aligned}
D_Y(H) &= c_1Y^5 + c_2Y^4Z + c_3Y^3Z^2 + c_4Y^2Z^3 + c_5YZ^4 + c_6Z^5 \\
&\leftrightarrow (c_1, c_2, c_3, c_4, c_5, c_6, 0)
\end{aligned}
$$

$$
\begin{aligned}
D_Z(H) &= c_2Y^5 + c_3Y^4Z + c_4Y^3Z^2 + c_5Y^2Z^3 + c_6YZ^4 + c_7Z^5 \\
&\leftrightarrow (c_2, c_3, c_4, c_5, c_6, c_7, 0).
\end{aligned}
$$

Hence

$$
\mathbf{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\
a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 1 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\
b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 \\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & 0 \\
c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & 0
\end{pmatrix}
$$

has rank 4 since $\mathbf{H}(R/I, 5) = 4$, and hence

$$
\mathbf{A_1} = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 \\
b_1 & b_2 & b_3 & b_4 & b_5 \\
b_2 & b_3 & b_4 & b_5 & b_6 \\
c_1 & c_2 & c_3 & c_4 & c_5 \\
c_2 & c_3 & c_4 & c_5 & c_6
\end{pmatrix}
$$

has rank 2.

We now consider the double contractions of F and G, where we identify a polynomial

$$
\alpha_1Y^4 + \alpha_2Y^3Z + \alpha_3Y^2Z^2 + \alpha_4YZ^3 + \alpha_5Z^4 + \beta XZ^3
$$

as a vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta) \in k^6$. Then we obtain

$$\begin{aligned}
D_{Z,X}(F) &= Z^4 \\
&\leftrightarrow (0,0,0,0,1,0) \\[1em]
D_{Y,Y}(F) &= a_1Y^4 + a_2Y^3Z + a_3Y^2Z^2 + a_4YZ^3 + a_5Z^4 \\
&\leftrightarrow (a_1, a_2, a_3, a_4, a_5, 0) \\[1em]
D_{Z,Y}(F) &= a_2Y^4 + a_3Y^3Z + a_4Y^2Z^2 + a_5YZ^3 + a_6Z^4 \\
&\leftrightarrow (a_2, a_3, a_4, a_5, a_6, 0) \\[1em]
D_{Z,Z}(F) &= a_3Y^4 + a_4Y^3Z + a_5Y^2Z^2 + a_6YZ^3 + a_7Z^4 + XZ^3 \\
&\leftrightarrow (a_3, a_4, a_5, a_6, a_7, 1) \\[1em]
D_{Y,Y}(G) &= b_1Y^4 + b_2Y^3Z + b_3Y^2Z^2 + b_4YZ^3 + b_5Z^4 \\
&\leftrightarrow (b_1, b_2, b_3, b_4, b_5, 0) \\[1em]
D_{Z,Y}(G) &= b_2Y^4 + b_3Y^3Z + b_4Y^2Z^2 + b_5YZ^3 + b_6Z^4 \\
&\leftrightarrow (b_2, b_3, b_4, b_5, b_6, 0) \\[1em]
D_{Z,Z}(G) &= b_3Y^4 + b_4Y^3Z + b_5Y^2Z^2 + b_6YZ^3 + b_7Z^4 \\
&\leftrightarrow (b_3, b_4, b_5, b_6, b_7, 0) \\[1em]
D_{Y,Y}(H) &= c_1Y^4 + c_2Y^3Z + c_3Y^2Z^2 + c_4YZ^3 + c_5Z^4 \\
&\leftrightarrow (c_1, c_2, c_3, c_4, c_5, 0) \\[1em]
D_{Z,Y}(H) &= c_2Y^4 + c_3Y^3Z + c_4Y^2Z^2 + c_5YZ^3 + c_6Z^4 \\
&\leftrightarrow (c_2, c_3, c_4, c_5, c_6, 0) \\[1em]
D_{Z,Z}(H) &= c_3Y^4 + c_4Y^3Z + c_5Y^2Z^2 + c_6YZ^3 + c_7Z^4 \\
&\leftrightarrow (c_3, c_4, c_5, c_6, c_7, 0).
\end{aligned}$$

Hence we have

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 & 1 \\ b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & 0 \\ b_3 & b_4 & b_5 & b_6 & b_7 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 \\ c_2 & c_3 & c_4 & c_5 & c_6 & 0 \\ c_3 & c_4 & c_5 & c_6 & c_7 & 0 \end{pmatrix}$$

has rank 6 since $\mathbf{H}(R/I, 4) = 6$, and so

$$\mathbf{B_1} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 \\ b_2 & b_3 & b_4 & b_5 \\ b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & c_5 \\ c_3 & c_4 & c_5 & c_6 \end{pmatrix}$$

has rank 4. Since the rank of $\mathbf{A}_1$ is 2, one of three row vectors of $\mathbf{A}_1$ has to be a linear combination of the other two.

Case 2–1. The row space of $\mathbf{A_1} = \langle (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle$.

Since

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4, c_5)(c_2, c_3, c_4, c_5, c_6) \\ \in \; & \langle (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6) \rangle, \end{aligned}$$

we have that

$$(a_1, a_2, a_3, a_4), (c_1, c_2, c_3, c_4), (c_2, c_3, c_4, c_5) \in \langle (b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5) \rangle$$

and

$$(a_2, a_3, a_4, a_5), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \in \langle (b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6) \rangle.$$

Then we obtain

$$\begin{aligned} & (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \\ \in \; & \langle (b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6) \rangle, \end{aligned}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 3, a contradiction.

One can show that the following **Case 2–2** is also impossible as **Case 2–1**.

Case 2–2. The row space of $\mathbf{A_1} = \langle (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6) \rangle$.

Case 2–3. The row space of $\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5) \rangle$.

Since

$$\begin{aligned} & (b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5) \rangle, \end{aligned}$$

we have that

$$(b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4), (c_2, c_3, c_4, c_5) \in \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$$

and

$$(b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \in \langle (a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5) \rangle.$$

Then we obtain

$$\begin{aligned} & (b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4) \rangle, \end{aligned}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 3, a contradiction.

We can also prove that the following case is impossible as **Case 2–3.**

Case 2–4. The row space of $\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4, c_5) \rangle$.

Case 2-5. The row space of $\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6) \rangle$.

Since

$$\begin{array}{rl} & (b_1, b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4, c_5), (c_2, c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6) \rangle, \end{array}$$

we have that

$$(b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4), (c_2, c_3, c_4, c_5) \in \langle (a_1, a_2, a_3, a_4), (b_2, b_3, b_4, b_5) \rangle$$

and

$$(b_2, b_3, b_4, b_5), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \in \langle (a_2, a_3, a_4, a_5), (b_3, b_4, b_5, b_6) \rangle.$$

Hence

$$\begin{array}{rl} & (b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (b_3, b_4, b_5, b_6) \rangle, \end{array}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 3, a contradiction.

Note that one can show that the following **Case 2–6** is also impossible as **Case 2–5.**

Case 2–6. The row space of $\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5), (c_2, c_3, c_4, c_5, c_6) \rangle$.

Case 2-7. The row space of $\mathbf{A_1} = \langle (a_1, a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4, c_5) \rangle$.

Since

$$\begin{array}{ll} & (b_1, b_2, b_3, b_4, b_5), (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4, c_5) \rangle, \end{array}$$

we have that

$$(b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5), (c_2, c_3, c_4, c_5) \in \langle (a_1, a_2, a_3, a_4), (c_1, c_2, c_3, c_4) \rangle$$

and

$$(b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6), (c_3, c_4, c_5, c_6) \in \langle (a_2, a_3, a_4, a_5), (c_2, c_3, c_4, c_5) \rangle.$$

Thus

$$\begin{array}{ll} & (b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \\ \in & \langle (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (c_1, c_2, c_3, c_4) \rangle, \end{array}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 3, a contradiction.

Case 2-8. The row space of $\mathbf{A_1} = \langle (b_1, b_2, b_3, b_4, b_5), (c_2, c_3, c_4, c_5, c_6) \rangle$.

Since

$$\begin{array}{rl} & (a_1, a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5) \\ \in & \langle (b_1, b_2, b_3, b_4, b_5), (c_2, c_3, c_4, c_5, c_6) \rangle \end{array}$$

we have that

$$(a_1, a_2, a_3, a_4), (b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4) \in \langle (b_1, b_2, b_3, b_4), (c_2, c_3, c_4, c_5) \rangle$$

and

$$(a_2, a_3, a_4, a_5), (b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5) \in \langle (b_2, b_3, b_4, b_5), (c_3, c_4, c_5, c_6) \rangle$$

Then we obtain

$$\begin{array}{l} (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5), (b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4), \\ (c_2, c_3, c_4, c_5) \in \langle (b_1, b_2, b_3, b_4), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6) \rangle, \end{array}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 3, a contradiction.

Note that one can show that the following **Case 2–9** is also impossible as **Case 2–8.**

Case 2–9. The row space of $\mathbf{A_1} = \langle (b_2, b_3, b_4, b_5, b_6), (c_1, c_2, c_3, c_4, c_5) \rangle$.

Case 2-10. The row space of $\mathbf{A_1} = \langle (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6)\rangle$.

Since

$$\begin{array}{ll} & (a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4, c_5) \\ \in & \langle (b_2, b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5, c_6)\rangle, \end{array}$$

we have that

$$(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \in \langle (b_2, b_3, b_4, b_5), (c_2, c_3, c_4, c_5)\rangle$$

and

$$(a_2, a_3, a_4, a_5), (b_2, b_3, b_4, b_5), (c_2, c_3, c_4, c_5) \in \langle (b_3, b_4, b_5, b_6), (c_3, c_4, c_5, c_6)\rangle.$$

Hence

$$\begin{array}{l} (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4), (b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4), \\ (c_2, c_3, c_4, c_5) \in \langle (b_3, b_4, b_5, b_6), (c_2, c_3, c_4, c_5), (c_3, c_4, c_5, c_6)\rangle, \end{array}$$

which means that the matrix $\mathbf{B_1}$ can have rank at most 3, a contradiction.

By **Cases 2–1$\sim$ 2–10**, $\mathbf{B_1}$ cannot have rank 4.

Hence, the O-sequence $\mathbf{H} = (1, 3, 4, 5, 6, 4, 3)$ cannot be level.

Using the same ideas as above, we can show that any O-sequence of the form

$$h = (1, 3, 4, 5, \ldots, 6, 4, 3)$$

cannot be level, as we wished. $\square$

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