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석 사 학 위 논 문

Some Construction of All Level  
Artinian  $O$ -sequences of Socle  
Degree 5 and type 5

2008

성신여자대학교 교육대학원  
교육학과 수학교육전공  
이 주 희

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이 논문을 석사학위논문으로 제출함.

2008 년 12 월

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# 인 준 서

이주희의 석사학위 논문으로 인준함.

심사위원 \_\_\_\_\_ 인

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## 논문개요

Froberg and Laksov 의 정리에 근거하여 여차원이 3, 형태가 5, 길이가 6 인 level 이 될 가능성이 있는 51 개의 Artinian O-Sequence 에 대하여 level 여부를 연구하였다.

우선 참고문헌 Generic Initials and Graded Artinian Level Algebras not having the Weak-Lefschetz Property, The Hilbert Function of a Level Algebra Memoris of the American Mathematical Society, Compressed Algebras, Complete Intersections 를 참고하여 level 이 될 가능성이 없는 19 개를 제외하였다.

먼저 24 개의 level 이 되는 수열은 CoCoA, S-plus 를 이용하여 “link-sum”을 구성하였다. 그리고 남은 8 개의 수열은 The Hilbert Function of a Level Algebra Memoris of the American Mathematical Society 을 참고하여 level 이 됨을 보였다. .

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ABSTRACT

## 1. Introduction

Let  $k$  be an infinite field,  $R = k[x_1, \dots, x_n] = \bigoplus_{i \geq 0} R_i$ , and let  $I$  be a homogeneous ideal of  $R$ ,  $A = R/I$ . The *Hilbert function of  $A$* ,  $\mathbf{H}_A : \mathbb{N} \rightarrow \mathbb{N}$  which is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t$$

where  $I$  is the ideal of a subscheme of  $\mathbb{P}^{n-1}$ , has been much studied.

Let  $\mathbf{H}$  be a Hilbert function of a finite set of points in  $\mathbb{P}^n$ . Then there are infinitely many different collections of points which share  $\mathbf{H}$  as their Hilbert function. In [6], they showed that there is a one to one correspondence between all possible Hilbert functions of finite sets of points in  $\mathbb{P}^n$  and  $n$ -type vectors (see Theorem 2.6, [6], see also [7] or [8]). Furthermore, in [2], [11] and [12], they showed that given  $\mathbf{H}$ , there is a unique maximal set of graded Betti numbers which can have.

In Section 2, we introduce some definitions and preliminary observations about level Artinian algebras (see also [9]). In [9], they used the fact that if a set of points in  $\mathbb{P}^2$  has level coordinate ring, then one can break the set up two subsets and used those subsets to construct a level algebra of codimension 3.

In Section 3, we prove that some special O-sequences cannot be level (see Proposition 3.5).

In Section 4, we introduce how to construct all codimension 3 level Artinian algebras of socle degree 5 and type 5. Moreover, we use computer programs ‘CoCoA’ and ‘Splus’ to construct all examples of level O-sequences.

## 2. Preliminaries

**Definition-Proposition 2.1** (Definition-Proposition 2.21, [9]). Let  $R = k[x_0, \dots, x_n]$  and let  $A = R/I$  be a Cohen-Macaulay ring of dimension  $d$ . Let

$$0 \rightarrow \mathcal{F}_{n-(d-1)} \rightarrow \cdots \rightarrow \mathcal{F}_1 \rightarrow I \rightarrow 0$$

be a minimal free resolution of  $I$ . Then

- (a) If  $B = B_0 \oplus \cdots \oplus B_\ell$  ( $B_\ell \neq 0$ ) is an Artinian algebra, then  $B$  is **level** if and only if  $B_\ell = \text{Ann}(B_1)$ .
- (b)  $A$  is a **level algebra** if  $\mathcal{F}_{n-(d-1)} = R^m(-s)$ , for some  $s > 0$ .  
rank  $\mathcal{F}_{n-(d-1)} = \text{Cohen-Macaulay type of } A$ .
- (c) i) If  $\mathbb{X}$  is a non-degenerate set of points in  $\mathbb{P}^n$ ,  $A = R/I_{\mathbb{X}}$  its coordinate ring, then we say that  $\ell$  is the **socle degree** of  $\mathbb{X}$  if  $\ell$  is the socle degree of the Artinian algebra  $B = A/\overline{L}A$ , where  $\overline{L}$  is any linear non-zero-divisor of  $A$ .
- ii)  $\mathbb{X}$  is called a **level set** of points if  $A = R/I_{\mathbb{X}}$  is a level algebra. In this case, the socle degree of  $\mathbb{X}$  is  $\ell = \sigma(\mathbb{X}) + n - 1$ .

- (d) If  $\bar{L}$  is a linear non-zero divisor in  $A = R/I$ , then  $A$  is level if and only if  $A/\bar{L}A \simeq A/(L, I_{\mathbb{X}})$  is level.
- (e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an Artinian algebra, see e.g., [9])  $b = \{b_i\}_{i \geq 0}$  with  $b_1 = n + 1$ , is called **level** if there is a level set of points in  $\mathbb{P}^n$  with Hilbert function  $b$ .

**Definition 2.2.** (a) A total order on the monomials of each degree

is said to be a **term order** if

- i)  $x_1 < \cdots < x_n$  or  $x_1 > \cdots > x_n$ , and
- ii)  $m_1 \geq m_2$  implies  $mm_1 \geq mm_2$ , for any monomials  $m, m_1$  and  $m_2$ .

(b) The **reverse lexicographic order** is a term order defined to be  $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$  if and only if

- i)  $\sum i_t > \sum j_t$  or
- ii)  $\sum i_t = \sum j_t$  and

there is  $s$  such that  $i_t = j_t$  for  $s < t \leq n$  and  $i_s < j_s$ .

(c) The **lexicographic order** is a term order defined to be  $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$  if and only if

- i)  $\sum i_t > \sum j_t$  or

ii)  $\sum i_t = \sum j_t$  and

there is  $s$  such that  $i_t = j_t$  for  $1 \leq t < s$  and  $i_s > j_s$ .

(d) Let  $S$  be a subset of all monomials in  $R_d$ .  $S$  is a **lex-segment** if a monomial  $m$  of degree  $d$  is in  $S$ , then every monomial  $m'$  of degree  $d$  in  $R_d$  such that  $m' > m$  is in  $S$ .

(e) Let  $I = \bigoplus_{t \geq 0} I_t$  be a graded ideal of  $R$ . We say that  $I$  is a **lex-segment ideal** if for every  $t \geq 0$ ,  $I_t$  is generated (as a vector space) by a lex-segment.

Here we have an interesting question.

**Question 2.3.** What are the possible Hilbert functions of level Artinian algebras of codimension 3 with  $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = 5$ ?

### 3. Non existence of Level O-sequences of Socle Degree 5 and Type 5

Before we construct all possible level Artinian O-sequences of socle degree 5 and type 5, we would like to introduce how to construct **all** Gorenstein O-sequences of socle degree 5 in the following Example 3.1 using Corollary 4.4. Notice that we used CoCoA [3], a system for

doing Computations in Commutative Algebra, to calculate the Hilbert functions in Example 3.1.

**Example 3.1.** Consider a compressed level O-sequence

$$\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 12 \ 6 \ 2.$$

Let  $\mathbb{X}$  be the set of all  $\bullet$ 's in  $\mathbb{Z}$  and  $\mathbb{Y}$  be the set of all  $*$ 's in  $\mathbb{Z}$ .

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & \bullet & \bullet \\ * & * & \bullet & \bullet \\ * & \bullet & * & \bullet \\ * & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} & : 1 \ 3 \ 6 \ 10 \ 13 \ 15 \ 16 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : 1 \ 3 \ 6 \ 6 \ 6 \ 6 \ 6 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : 1 \ 3 \ 6 \ 6 \ 3 \ 1 \ 0 \ \rightarrow . \end{aligned}$$

We recall a special case of a theorem of Fröberg and Laksov of [4] that if  $A$  is a level Artinian algebra of codimension 3 and type  $m$  with  $\sigma - 1 = t$ , then

$$\mathbf{H}(A, i) \leq \min \left\{ \binom{2+i}{2}, m \binom{t-i+2}{2} \right\}.$$

The following algorithm for CoCoA is due to Georgio Daldozzo in Italy.

Define HvectorAdmis2(K)

L := [[1,3]];

For I:=2 To K Do

Prov := [];

```

Foreach El In L[I-1] Do
  For J:=1 To BinExp(El[Len(El)],I-1,1,1) Do
  Append(Prov,Concat(El,[J]));
  EndFor;
EndForeach;
Append(L,Prov);
EndFor;
A := L[Len(L)];
B:= [Elements In A | Elements[Len(Elements)]=5];
Return [Elements In B | Elements[Len(Elements)-1]< 6];
EndDefine;

```

Before we find all possible level Artinian O-sequences of socle degree 5 and type 5, we need the following theorems and corollary to eliminate some of non-level cases.

Before we find all possible level Artinian O-sequences of socle degree 5 and type 4, we need the following theorems and corollary to eliminate some of non-level cases.

**Theorem 3.2** (Theorem 2.12, [5]). *Let  $h_{d-2}$ ,  $h_{d-1}$ ,  $h_d$  be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d-1 \rangle} \quad \text{and} \quad h_{d-1} = h_{d-2}^{\langle d-2 \rangle}.$$

1)	1, 3, 3, 4, 5, 5	2)	1, 3, 4, 4, 5, 5	3)	1, 3, 4, 5, 5, 5
4)	1, 3, 4, 5, 6, 5	5)	1, 3, 5, 4, 5, 5	6)	1, 3, 5, 5, 5, 5
7)	1, 3, 5, 5, 6, 5	8)	1, 3, 5, 6, 5, 5	9)	1, 3, 5, 6, 6, 5
10)	1, 3, 5, 6, 7, 5	11)	1, 3, 5, 7, 5, 5	12)	1, 3, 5, 7, 6, 5
13)	1, 3, 5, 7, 7, 5	14)	1, 3, 5, 7, 8, 5	15)	1, 3, 5, 7, 9, 5
16)	1, 3, 6, 4, 5, 5	17)	1, 3, 6, 5, 5, 5	18)	1, 3, 6, 5, 6, 5
19)	1, 3, 6, 6, 5, 5	20)	1, 3, 6, 6, 6, 5	21)	1, 3, 6, 6, 7, 5
22)	1, 3, 6, 7, 5, 5	23)	1, 3, 6, 7, 6, 5	24)	1, 3, 6, 7, 7, 5
25)	1, 3, 6, 7, 8, 5	26)	1, 3, 6, 7, 9, 5	27)	1, 3, 6, 8, 5, 5
28)	1, 3, 6, 8, 6, 5	29)	1, 3, 6, 8, 7, 5	30)	1, 3, 6, 8, 8, 5
31)	1, 3, 6, 8, 9, 5	32)	1, 3, 6, 8, 10, 5	33)	1, 3, 6, 9, 5, 5
34)	1, 3, 6, 9, 6, 5	35)	1, 3, 6, 9, 7, 5	36)	1, 3, 6, 9, 8, 5
37)	1, 3, 6, 9, 9, 5	38)	1, 3, 6, 9, 10, 5	39)	1, 3, 6, 9, 11, 5
40)	1, 3, 6, 9, 12, 5	41)	1, 3, 6, 10, 5, 5	42)	1, 3, 6, 10, 6, 5
43)	1, 3, 6, 10, 7, 5	44)	1, 3, 6, 10, 8, 5	45)	1, 3, 6, 10, 9, 5
46)	1, 3, 6, 10, 10, 5	47)	1, 3, 6, 10, 11, 5	48)	1, 3, 6, 10, 12, 5
49)	1, 3, 6, 10, 13, 5	50)	1, 3, 6, 10, 14, 5	51)	1, 3, 6, 10, 15, 5

TABLE 1. All possible Artinian O-sequences of length 6 and type 5

Let  $I$  be any ideal in  $R = k[x_1, \dots, x_n]$  such that the Hilbert function of  $R/I$  satisfies

$$\begin{aligned} \mathbf{H}(R/I, d-2) &= h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) &= h_{d-1} \\ \mathbf{H}(R/I, d) &= h_d. \end{aligned}$$

Then, the ring  $R/I$  has socle of dimension  $\varepsilon$  in degree  $d-2$ .

**Theorem 3.3** (Theorem 8, [5]). *If  $\mathbf{H} : h_0 \ h_1 \ \cdots \ h_t \ 0 \rightarrow$  is a level Artinian O-sequence where  $t \geq 2$ , then  $\mathbf{G} : h_0 \ h_1 \ \cdots \ h_{t-1} \ 0 \rightarrow$  is also a level Artinian O-sequence.*

Using Theorems 3.3 and 3.4, the following 7 cases in Table 2 are not level.

1)	1, 3, 3, 4, 5, 5	2)	1, 3, 4, 4, 5, 5	5)	1, 3, 5, 4, 5, 5
7)	1, 3, 5, 5, 6, 5	16)	1, 3, 6, 4, 5, 5	18)	1, 3, 6, 5, 6, 5
26)	1, 3, 6, 7, 9, 5				

TABLE 2. By Theorems 3.3 and 3.4, the 7 cases are not level.

**Theorem 3.4** (Theorem 4.1, [1]). *Let  $R = k[x_1, x_2, x_3]$  and let  $\mathbf{H} = (h_0, h_1, \dots, h_s)$  be the  $h$ -vector of a graded Artinian algebra  $A = R/I$  with socle degree  $s$ . If*

$$h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \leq 2d + 3,$$

*then  $\mathbf{H}$  is not level*

By Theorem 3.5, the following 8 cases in Table 3 are not level.

8)	1, 3, 5, 6, 5, 5	11)	1, 3, 5, 7, 5, 5	17)	1, 3, 6, 5, 5, 5
19)	1, 3, 6, 6, 5, 5	22)	1, 3, 6, 7, 5, 5	27)	1, 3, 6, 8, 5, 5
33)	1, 3, 6, 9, 5, 5	41)	1, 3, 6, 10, 5, 5		

TABLE 3

**Proposition 3.5.** *There is no level Artinian algebra with Hilbert function  $h = (1, 3, 6, 8, 6, 5)$ .*

```

Define ADDUPHilbert(L)
  S:=[];
  For I := 2 To Len(L)
    Do S1:=Sum(First(L,I));
    Append(S,S1);
  EndFor;
  S2:=Comp(S,Len(S));
  S:=[S];
  Append(S,S2);
  S;
EndDefine;

ADDUPHilbert([1,3,6,8,6,5]); [[4, 10, 18, 24, 29], 29]
-----
TV.FromHF([[4, 10, 18, 24, 29], 29]); [[[2]], [[1], [2],
[4]], [[2], [3], [4], [5], [6]]]
-----
TV.FromHF([[4, 10, 18, 24, 29], 29]); [[[2]], [[1], [2],
[4]], [[2], [3], [4], [5], [6]]]
-----

```

*Proof.* Consider the  $h$ -vector  $h = (1, 3, 6, 8, 6, 5)$  Hilbert function. The minimal free resolution of the lex-segment ideal  $I^{\text{lex}}$  associated to  $I$  is

$$\begin{aligned} 0 &\rightarrow \mathbf{R}^3(-6) \oplus R(-7) \oplus R^5(-8) \\ &\rightarrow R(-4) \oplus R^7(-5) \oplus \mathbf{R}^3(-6) \oplus R^{10}(-7) \\ &\rightarrow R^2(-3) \oplus R^4(-4) \oplus R^2(-5) \oplus R^5(-6) \rightarrow R \rightarrow R/I^{\text{lex}} \rightarrow 0 \end{aligned}$$

and suppose that  $A = R/I$ ,  $R = k[x_1, x_2, x_3]$  is a level algebra with  $h$ .

*Claim :* The minimal number of generators of  $I$ , in degree 6, is  $< 4$ .

Notice that once this claim is proved we are done. Since then  $\beta_{0,6}(A) < 5$  and by the Bigatti-Hulett-Pardue ([2],[11],[12]) result we must have  $\beta_{1,6}(A) < 3$ . This, in turn, implies that  $\beta_{2,6}(A) \geq 1$  and hence  $A$  cannot be level.

*Proof of Claim.* Suppose that  $I$  is a level ideal with Hilbert function  $(1,3,6,8,6,5)$  and  $I$  has 4 generators in degree 6 and let  $J = (I_{\leq 5})$ . Then the Hilbert function of  $R/J$  begins 1 3 6 8 6 5 5  $\cdots$ . Now, we can use Theorem 3.4 in [4] to assert that  $R/J$  has at least 1-dimensional socle in degree 4. Since  $R/J$  and  $R/I$  agree in degree  $\leq 5$  we conclude that  $R/I$  cannot be level, a contradiction, as we wished.

□

**Example 3.6.** Let  $I_1, I_2,$  and  $I_3$  be ideals of  $R = k[x_1, x_2, x_3]$  such that Hilbert functions of  $R/I_1, R/I_2,$  and  $R/I_3$  are  $(1, 3, 6, 6, 7, 5),$   $(1, 3, 6, 9, 6, 5),$  and  $(1, 3, 6, 10, 6, 5)$  respectively.

Then the Minimal free resolutions of  $R/I_1, R/I_2$  and  $R/I_3$  are

$$\begin{aligned} 0 &\rightarrow \mathbf{R}(-5) \oplus R^2(-7) \oplus R^5(-8) \rightarrow R^4(-4) \oplus R^5(-6) \oplus R^{10}(-7) \\ &\rightarrow R^4(-3) \oplus R^3(-5) \oplus R^5(-6) \rightarrow R \rightarrow R/I_1^{\text{lex}} \rightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 &\rightarrow \mathbf{R}^4(-6) \oplus R(-7) \oplus R^5(-8) \rightarrow R^{10}(-5) \oplus \mathbf{R}^3(-6) \oplus R^{10}(-7) \\ &\rightarrow R(-3) \oplus R^6(-4) \oplus R^2(-5) \oplus R^5(-6) \rightarrow R \rightarrow R/I_2^{\text{lex}} \rightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 &\rightarrow \mathbf{R}^5(-6) \oplus R(-7) \oplus R^5(-8) \rightarrow R^{13}(-5) \oplus \mathbf{R}^3(-6) \oplus R^{10}(-7) \\ &\rightarrow R^9(-4) \oplus R^2(-5) \oplus R^5(-6) \rightarrow R \rightarrow R/I_3^{\text{lex}} \rightarrow 0, \end{aligned}$$

Respectively. As we see from the above minimal free resolutions of all three cases last free modules have non-cancelable shifts, and hence they cannot be level.

#### 4. Some Construction of Gorenstein and Level Artinian

##### O-sequences

**Definition 4.1** (Definition 3.1, [10]). (a) A finite set  $\mathbb{X}$  of points

$\mathbb{P}^2$  is called a *basic configuration* of type  $(d, e)$  if there exist

distinct elements  $b_j, c_j$  in  $k$  such that

$$I_{\mathbb{X}} = \left( \prod_{j=1}^d (x - b_j z), \prod_{j=1}^e (y - c_j z) \right).$$

We denote  $\mathbb{X} := \mathbb{B}(d, e)$ .

(b) A finite set  $\mathbb{X}$  of points in  $\mathbb{P}^2$  is called a ***pure configuration*** if there exist finite basic configurations  $\mathbb{B}(d_1, e_1), \dots, \mathbb{B}(d_m, e_m)$  where  $e_1 > \dots > e_m$ , which satisfy the following three conditions:

- (i)  $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \emptyset$  if  $i \neq j$ ,
- (ii)  $\mathbb{X} = \mathbb{B}(d_1, e_1) \cup \dots \cup \mathbb{B}(d_m, e_m)$ ,
- (iii)  $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$  for all  $1 \leq i \leq m-1$ , where  $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \rightarrow \mathbb{P}^1$  is the map defined by sending the point  $(x, y, z)$  to the point  $(y, z)$ . In this case, we denote  $\mathbb{X} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ .

**Proposition 4.2** (Corollary 3.10, [9]). *Let  $\mathbb{X} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$  be a pure configuration in  $\mathbb{P}^2$ . Then  $\mathbb{X}$  is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all  $1 \leq i \leq m-1$ .

If  $r(A)$  is Cohen-Macaulay type of a Cohen-Macaulay standard graded  $k$ -algebra  $A$ , and if  $A = \bigoplus_{i \geq 0} A_i$  is an Artinian level algebra, then

$r(A) = \dim_k A_{\sigma(A)-1}$ , where  $\sigma(A) = \min\{i | A_i = 0\}$ . If  $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$  is a level pure configuration in  $\mathbb{P}^2$ , then  $r(\mathbb{Z}) = r(R/I_{\mathbb{Z}}) = m$ .

**Lemma 4.3** (Lemma 3.14, [9]). *Let  $\mathbb{Z}$  be a level set of points in  $\mathbb{P}^n$  and  $\mathbb{X}$  a subset of  $\mathbb{Z}$ . Set  $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$ . Then  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  is an Artinian level graded  $k$ -algebra with  $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \sigma(\mathbb{Z}) - 1$  and  $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq r(\mathbb{Z})$ .*

**Corollary 4.4** (Corollary 3.15, [9]). *Let  $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$  be a level pure configuration in  $\mathbb{P}^2$  and  $\mathbb{X}$  a subset of  $\mathbb{Z}$ . Set  $\mathbb{Y} := \mathbb{Z} \setminus \mathbb{X}$ . Then  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  is an Artinian level graded  $k$ -algebra with  $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = d_1 + e_1 - 2$  and  $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq m$ .*

Now we classify all cases in Table 1 in the following example.

**Example 4.5.** Now we construct 24 level Artinian O-sequences among 51 cases in Table 1, which are: 3), 4), 6), 9), 10), 13), 14), 15), 20), 23), 24), 25), 29), 30), 31), 32), 36), 37), 38), 39), 44), 45), 46), 47).

3)

$$\mathbb{Z} = \left\{ \begin{array}{cccccc} \bullet & & & & & \\ * & * & & & & \\ \bullet & * & * & & & \\ * & * & * & * & & \\ \bullet & * & * & * & * & \\ \bullet & * & * & * & \bullet & \\ * & * & * & * & * & \end{array} \right.$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 4 \ 5 \ 5 \ 5 \ 5 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 20 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 4 \ 5 \ 5 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

4)

$$\mathbb{Z} = \begin{cases} \bullet \\ \bullet \ * \\ * \ * \ * \\ \bullet \ * \ * \ \bullet \\ * \ * \ * \ * \ * \\ \bullet \ * \ * \ * \ * \\ \bullet \ * \ * \ * \ * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 4 \ 5 \ 6 \ 6 \ 6 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 19 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 4 \ 5 \ 6 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

6)

$$\mathbb{Z} = \begin{cases} \bullet \\ * \ * \\ * \ * \ * \\ * \ \bullet \ * \ * \\ * \ * \ * \ * \ \bullet \\ * \ * \ * \ * \ \bullet \\ * \ \bullet \ * \ * \ * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 5 \ 5 \ 5 \ 5 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 20 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 5 \ 5 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

9)

$$\mathbb{Z} = \begin{cases} \bullet \\ * & \bullet \\ \bullet & * & * \\ * & * & * & * \\ \bullet & * & * & * & * \\ \bullet & * & * & * & * \\ * & \bullet & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 20 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 5 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 19 & 19 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 5 & 6 & 6 & 5 & 0 & \rightarrow . \end{aligned}$$

10)

$$\mathbb{Z} = \begin{cases} * \\ \bullet & * \\ * & * & * \\ * & * & * \\ * & * & * & * & * \\ * & \bullet & * & * & * \\ \bullet & * & \bullet & \bullet & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 26 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 5 & 6 & 7 & 8 & 8 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 18 & 18 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 5 & 6 & 7 & 5 & 0 & \rightarrow . \end{aligned}$$

13)

$$\mathbb{Z} = \begin{cases} \bullet \\ \bullet & \bullet \\ \bullet & \bullet & * \\ * & * & * & * \\ * & \bullet & * & * & * \\ * & * & * & * & * \\ * & \bullet & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned}
 \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\
 \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 7 \ 7 \ 7 \ 7 \ \rightarrow \\
 \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 18 \ 18 \ \rightarrow \\
 \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 7 \ 7 \ 5 \ 0 \ \rightarrow .
 \end{aligned}$$

14)

$$\mathbb{Z} = \begin{cases} \bullet & * & * & & & & & \\ * & \bullet & * & * & & & & \\ * & * & \bullet & * & * & & & \\ \bullet & \bullet & * & \bullet & * & \bullet & & \\ * & * & * & * & \bullet & * & * & \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned}
 \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\
 \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ 8 \ \rightarrow \\
 \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 17 \ 17 \ \rightarrow \\
 \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 7 \ 8 \ 5 \ 0 \ \rightarrow .
 \end{aligned}$$

20)

$$\mathbb{Z} = \begin{cases} \bullet & & & & & & & \\ * & * & & & & & & \\ * & * & * & & & & & \\ * & * & \bullet & \bullet & & & & \\ * & * & * & * & * & & & \\ \bullet & * & * & \bullet & * & & & \\ * & \bullet & * & * & * & & & \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned}
 \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\
 \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 6 \ 6 \ 6 \ 6 \ \rightarrow \\
 \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 19 \ \rightarrow \\
 \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 6 \ 6 \ 5 \ 0 \ \rightarrow .
 \end{aligned}$$

23)

$$\mathbb{Z} = \begin{cases} * \\ * & * \\ * & * & \bullet \\ * & \bullet & * & * \\ \bullet & * & * & * & \bullet \\ \bullet & * & * & * & \bullet \\ * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 20 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 14 & 18 & 18 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 7 & 6 & 5 & 0 & \rightarrow . \end{aligned}$$

24)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ * & \bullet & * \\ \bullet & \bullet & * & * \\ * & * & * & \bullet & * \\ \bullet & * & * & * & * & * \\ \bullet & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 26 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 19 & 19 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 7 & 7 & 5 & 0 & \rightarrow . \end{aligned}$$

25)

$$\mathbb{Z} = \begin{cases} * & \bullet \\ * & \bullet & * \\ * & * & * & \bullet \\ * & * & * & * & * \\ * & * & * & * & * & * \\ \bullet & \bullet & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 26 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 7 \ 8 \ 8 \ 8 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 18 \ 18 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 7 \ 8 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

29)

$$\mathbb{Z} = \begin{cases} * \\ * \ * \\ \bullet \ * \ \bullet \\ \bullet \ * \ * \ * \\ * \ * \ * \ * \ \bullet \\ * \ \bullet \ * \ * \ \bullet \\ * \ \bullet \ * \ * \ \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 8 \ 8 \ 8 \ 8 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 14 \ 17 \ 17 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 8 \ 7 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

30)

$$\mathbb{Z} = \begin{cases} \bullet \ * \\ * \ * \ \bullet \\ \bullet \ * \ * \ * \\ \bullet \ * \ \bullet \ * \ * \\ * \ * \ * \ \bullet \ * \ * \\ * \ * \ * \ \bullet \ \bullet \ * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 26 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 8 \ 8 \ 8 \ 8 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 18 \ 18 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 8 \ 8 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

31)

$$\mathbb{Z} = \begin{cases} \bullet \\ * & \bullet \\ * & * & * \\ * & * & * & \bullet \\ * & * & * & * & \bullet \\ * & * & \bullet & * & * \\ \bullet & * & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 20 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 8 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 8 & 9 & 5 & 0 & \rightarrow . \end{aligned}$$

32)

$$\mathbb{Z} = \begin{cases} * & * \\ * & \bullet & \bullet \\ * & * & \bullet & * \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ * & * & \bullet & \bullet & \bullet & * \\ \bullet & * & \bullet & \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 20 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 10 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 8 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 8 & 10 & 5 & 0 & \rightarrow . \end{aligned}$$

36)

$$\mathbb{Z} = \begin{cases} * \\ * & * \\ \bullet & * & * \\ * & \bullet & \bullet & * \\ * & \bullet & * & * & \bullet \\ * & \bullet & * & * & \bullet \\ * & * & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 14 \ 16 \ 16 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 9 \ 8 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

37)

$$\mathbb{Z} = \begin{cases} * \\ * \bullet \\ * * * \\ \bullet * * \bullet \\ \bullet \bullet \bullet \bullet * \\ \bullet \bullet * * * \\ * * * * * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 16 \ 16 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 9 \ 9 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

38)

$$\mathbb{Z} = \begin{cases} \bullet \\ \bullet * \\ \bullet * * \\ * * * * \\ \bullet \bullet * * * \\ * * \bullet * * \\ * \bullet \bullet \bullet \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 9 \ 10 \ 10 \ 10 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 15 \ 15 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 9 \ 10 \ 5 \ 0 \ \rightarrow . \end{aligned}$$

39)

$$\mathbb{Z} = \begin{cases} \bullet \\ \bullet & * \\ * & \bullet & * \\ * & * & \bullet & \bullet \\ * & * & \bullet & \bullet \\ * & \bullet & \bullet & * & \bullet & * \\ \bullet & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 26 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 9 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 9 & 11 & 5 & 0 & \rightarrow . \end{aligned}$$

44)

$$\mathbb{Z} = \begin{cases} \bullet \\ * & * \\ * & \bullet & * \\ * & * & \bullet & \bullet \\ * & * & * & \bullet & \bullet \\ * & * & * & \bullet & * \\ * & \bullet & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 20 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 13 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 10 & 8 & 5 & 0 & \rightarrow . \end{aligned}$$

45)

$$\mathbb{Z} = \begin{cases} * \\ * & * \\ * & \bullet & \bullet \\ * & \bullet & * & * \\ \bullet & \bullet & \bullet & \bullet & * \\ * & * & \bullet & \bullet & * \\ * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned}
 \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\
 \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ \rightarrow \\
 \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 14 \ 15 \ 15 \ \rightarrow \\
 \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 10 \ 9 \ 5 \ 0 \ \rightarrow .
 \end{aligned}$$

46)

$$\mathbb{Z} = \begin{cases} * \\ \bullet \ * \\ \bullet \ * \ \bullet \\ * \ * \ * \ \bullet \\ \bullet \ * \ * \ * \ \bullet \\ * \ \bullet \ * \ \bullet \ \bullet \\ * \ * \ \bullet \ * \ * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned}
 \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 25 \ \rightarrow \\
 \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ \rightarrow \\
 \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 15 \ 15 \ \rightarrow \\
 \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 10 \ 10 \ 5 \ 0 \ \rightarrow .
 \end{aligned}$$

47)

$$\mathbb{Z} = \begin{cases} \bullet \\ * \ * \\ * \ * \ * \\ \bullet \ * \ \bullet \ * \\ * \ \bullet \ \bullet \ * \\ * \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \\ * \ * \ * \ * \ * \ \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned}
 \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 26 \ \rightarrow \\
 \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 10 \ 11 \ 11 \ 11 \ \rightarrow \\
 \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 15 \ 15 \ \rightarrow \\
 \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 10 \ 11 \ 5 \ 0 \ \rightarrow .
 \end{aligned}$$

**Example 4.6.** Note that  $(1,3,5,7,6,5,2)$  is level (See Appendix B in [5]). Hence, by Theorem 3.4, the case  $12)(1,3,5,7,6,5,)$  is also level.

**Theorem 4.7** (Proposition 5.24, [5]). *Let  $(1, h_1, h_2, \dots, h_s)$  be the  $h$ -vector of a level algebra  $A = R/I$  where  $R = k[x, y, z]$ . Let  $j = \max\{i \mid h_i = \binom{2+i}{i}\}$ . Then, there is an ideal  $J \subset I$  so that  $B = R/J$  is a level algebra with  $h$ -vector  $(1, h_1, \dots, h_j, h_{j+1} + 1, \dots, h_s + 1)$ .*

Consider an O-sequence  $43)(1, 3, 6, 10, 7, 5)$  Note that  $(1, 3, 6, 9, 6, 4)$  is level.(See Appendix C in [5]).

Applying Theorem 4.7,  $(1,3,6,9+1,6+1,4+1)=(1,3,6,10,7,5)$  is also level, i.e, the case 43)  $(1,3,6,10,7,5)$  is level, as we wanted.

**Proposition 4.8** (Remark 5.32, [5]). *The link-sum construction cannot be used to construct level Artinian algebras of type 5 in codimension 3 with Hilbert function  $(1, 3, \alpha, \beta, \gamma, 5)$  and  $12 \leq \gamma \leq 15$  from level sets of points in  $\mathbb{P}^2$ .*

*Proof.* We shall show that  $(1, 3, \alpha, \beta, s, 5)$  cannot be constructed for  $12 \leq s \leq 15$ , using level sets of points in  $\mathbb{P}^2$ . Suppose we had a level set of points  $\mathbb{Z}$  in  $\mathbb{P}^2$  and  $\mathbb{Z} = \mathbb{X} \cup \mathbb{Y}$  which gave Hilbert Function as

above by the link-sum construction. Then if we let  $A = R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ ,

we would have a display

$$\begin{array}{rccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & - & - & b & c & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & - & - & \alpha_1 & \alpha_2 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & - & - & \beta_1 & \beta_2 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & - & - & s & 5 & 0 \quad . \end{array}$$

where  $12 \leq s \leq 15$ .

The construction method says that  $b + s = \alpha_1 + \beta_1$  and  $c + 4 = \alpha_2 + \beta_2$ .

By subtracting, we have

$$(c - b) + (5 - s) = (\alpha_2 - \alpha_1) + (\beta_2 - \beta_1).$$

But, since  $\mathbb{X}$  and  $\mathbb{Y}$  are point sets in  $\mathbb{P}^2$  we have  $(\alpha_2 - \alpha_1) \geq 0$  and  $(\beta_2 - \beta_1) \geq 0$ . Since  $5 - s \leq -7$  we must have  $c - b \geq 7$ . But, since  $\mathbb{Z}$  has a differentiable Hilbert function, the maximum value for  $c - b$  is  $6 = \dim_k k[x, y]_5$ , which is a contradiction, as we wished.  $\square$

By Proposition 4.8, the following cases in Table 4 cannot be obtained using the link-sum construction.

40)	1, 3, 6, 9, 12, 5	48)	1, 3, 6, 10, 12, 5
49)	1, 3, 6, 10, 13, 5	50)	1, 3, 6, 10, 14, 5
51)	1, 3, 6, 10, 15, 5		

TABLE 4

However, in [5], they showed that above O-sequence in Table 4 are all level (See Appendix C in [5]).

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# ABSTRACT

## Some Construction of All Level Artinian O-sequences of Socle Degree 5 and type 5

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The purpose of this study was to examine the Codimension 3, Socle Degree 5 and Type 5 Artinian O-sequences 51 based on Theorem of Froberg and Laksov.

First of all, we prove that 19 sequences cannot be level using [1], [4], [6].

Then we introduce how to construct 24 sequences using CoCoA, S-plus. And we prove that 8 sequences can be level by Theorem in [6].