신 용 수 교수지도

석 사 학 위 논 문

Some Construction of All Level Artinian O-sequences of Socle Degree 5 and Type 4

2008

성신여자대학교 교육대학원

교육학과 수학교육전공

鄭 洗 伶

Some Construction of All Level Artinian O-sequences of Socle Degree 5 and Type 4

신 용 수 교수지도

이 논문을 석사학위논문으로 제출함.

2008월 5월

성신여자대학교 교육대학원

교육학과 수학교육전공

鄭 洗 伶

인 준 서

鄭洗伶의 석사학위 논문으로 인준함.

심사위원 ________________ 인

심사위원 ________________ 인

심사위원 ________________ 인

성신여자대학교 교육대학원

논 문 개 요

Fröberg and Laksov의 정리에 근거하여 여차원이 3, 형태가 4, 길이가 6인 level이 될 가능성이 있는 62개의 Artinian O-Sequence에 대하여 level 여부를 연구하였다.

우선 참고문헌 Generic Initials and Graded Artinian Level Algebras not having the Weak-Lefschetz Property, The Hilbert Funtion of a Level Algebra Memoris of the American Mathemetical Society, Compressed Algebras, Complete Intercections를 참고하여 level이 될 가능성이 없는 28개를 제외하였다.

먼저 29개의 level이 되는 수열은 CoCoA, S-plus를 이용하여 "link-sum"을 구성하였다. 그리고 남은 5개의 수열은 The Hilbert Funtion of a Level Algebra Memoris of the American Mathemetical Society을 참고하여 level이 됨을 보였다.

목 차

1. **Introduction**

Let k be an infinite field, $R = k[x_1, \ldots, x_n] = \bigoplus_{i \geq 0} R_i$, and let I be a homogeneous ideal of R, $A = R/I$. The ***Hilbert function of*** A, $\mathbf{H}_A : \mathbb{N} \longrightarrow \mathbb{N}$, defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t$$

where I is the ideal of a subscheme of $\mathbb{P}^{n-1}$, contains a great deal of information about the geometry of this subscheme.

Let $\mathbf{H}$ be a numerical function which can be the Hilbert function of a finite set of points in $\mathbb{P}^n$. For any particular $\mathbf{H}$, there are infinitely many different collections of points which share $\mathbf{H}$ as their Hilbert function. In [7], they showed that there is a one to one correspondence between all possible Hilbert functions of finite sets of points in $\mathbb{P}^n$ and n-type vectors (see Theorem 2.6, [7], see also [8] or [9]).

In [6], they asked the following questions.

(1) If $\mathbf{H}$ is a possible Hilbert function for a set of reduced points in $\mathbb{P}^n$, what are all the possible graded Betti numbers for sets of points $\mathbb{X} \subseteq \mathbb{P}^n$ which have Hilbert function $\mathbf{H}$?

(2) What can be the Hilbert function of a level Artinian algebra (see Section 2 for the definition)?

By works of Bigatti [2], Hulett [12], and Pardue [13], we know that, given $\mathbf{H}$, there is a unique maximal set of graded Betti numbers which can have.

In Section 2, we introduce some definitions and preliminary observations about Gorenstein and level Artinian algebras (see also [10]). In [10], they used the fact that if a set of points in $\mathbb{P}^2$ has level coordinate ring, then one can break the set up two subsets and used those subsets to construct a level algebra of codimension 3. Moreover, they used this method to get a complete list (in codimension 3) of all Hilbert functions of level algebras of type 2 and socle degree ≤ 5. Recently, in [6], they obtained all possible Hilbert functions (in codimension 3) of level algebras of socle degree 6 and type 2.

In Section 3, we prove that some sepecial O-sequences cannot be level (see Proposition 3.6)

In Section 4, we introduce how to construct all codimension 3 level Artinian algebras of socle degree 5 and type 4.

2. **Preliminary Definitions and Results**

Definition-Proposition 2.1 (Definition-Proposition 2.21, [10])**.** Let $R = k[x_0, \dots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. Then

(a) If $B = B_0 \bigoplus \cdots \bigoplus B_\ell$ $(B_\ell \neq 0)$ is an Artinian algebra, then B is ***level*** if and only if $B_\ell = \operatorname{Ann}(B_1)$.

(b) A is a ***level algebra*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. $\operatorname{rank} \mathcal{F}_{n-(d-1)} =$ Cohen-Macaulay type of A.

(c) i) If $\mathbb{X}$ is a non-degenerate set of points in $\mathbb{P}^n$, $A = R/I_{\mathbb{X}}$ its coordinate ring, then we say that ℓ is the ***socle degree*** of $\mathbb{X}$ if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where $\overline{L}$ is any linear non-zero-divisor of A.

ii) $\mathbb{X}$ is called a ***level set*** of points if $A = R/I_{\mathbb{X}}$ is a level algebra. In this case, the socle degree of $\mathbb{X}$ is $\ell = \sigma(\mathbb{X}) + n - 1$.

(d) If $\overline{L}$ is a linear non-zero divisor in $A = R/I$, then A is level if and only if $A/\overline{L}A \simeq A/(L, I_{\mathbb{X}})$ is level.

(e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an Artinian algebra, see e.g., [9]) $b = \{b_i\}_{i \geq 0}$ with $b_1 = n + 1$, is called ***level*** if there is a level set of points in $\mathbb{P}^n$ with Hilbert function b.

Definition 2.2. (a) A total order on the monomials of each degree is said to be a ***term order*** if

i) $x_1 > \cdots > x_n$, and

ii) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m, m_1 and m_2.

(b) The ***reverse lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $s < t \leq n$ and $i_s < j_s$.

(c) The ***lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $1 \leq t < s$ and $i_s > j_s$.

(d) Let S be a subset of all monomials in R_d. S is a ***lex-segment*** if a monomial m of deree d is in S, then every monomial m' of degree d in R_d such that $m' > m$ is in S.

(e) Let $I = \bigoplus_{t \geq 0} I_t$ be a graded ideal of R. We say that I is a ***lex-segment ideal*** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

Here we have an interesting question.

Question 2.3. What are the possible Hilbert functions of level Artinian algebras of codimension 3 with $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = 4$?

3. Non existence of Level O-sequences of Socle Degree 5 and Type 4

Before we construct all possible level Artinian O-sequences of socle degree 5 and type 4, we would like to introduce how to construct ***all*** Gorenstein O-sequences of socle degree 5 in the following Example 3.1 using Corollary 4.6. Notice that we used CoCoA [3], a system for doing **Co**mputations in **Co**mmutative **A**lgebra, to calculate the Hilbert functions in Example 3.1.

Example 3.1. Consider any level O-sequence $\mathbf{H} : 1 \quad 3 \quad \alpha \quad \beta \quad 3 \quad 1 \quad 0$. Since the sequence $\mathbf{H}$ is of type 1, we have that $\mathbf{H}$ is a Gorenstein sequence of socle degree 5, and hence we have that

$$\alpha = \beta = \ 3,\ 4,\ 5 \ \text{ or } \ 6.$$

Let $\mathbb{X}$ be the set of all $\bullet$'s in $\mathbb{Z}$ and $\mathbb{Y}$ be the set of all $*$'s in $\mathbb{Z}$.

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & \bullet & \bullet \\ * & * & \bullet & \bullet \\ * & \bullet & * & \bullet \\ * & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 13 & 15 & 16 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 6 & 3 & 1 & 0 & \rightarrow . \end{array}$$

By the same idea, we can construct all the other cases when $\alpha = \beta = 3,\ 4,\ \text{or},\ 5$.

We recall a special case of a theorem of Fröberg and Laksov of [4] that if A is a level Artinian algebra of codimension 3 and type m with $\sigma - 1 = t$, then

$$\mathbf{H}(A,i) \leq \min\left\{\binom{2+i}{2}, m\binom{t-i+2}{2}\right\}.$$

The following algorithm for CoCoA is due to Georgio Daldozzo in Italy.

```
Define HvectorAdmis2(K)
  L := [[[1,3]]];
  For I:=2 To K Do
    Prov := [];
    Foreach El In L[I-1] Do
      For J:=1 To BinExp(El[Len(El)],I-1,1,1) Do
    Append(Prov,Concat(El,[J]));
      EndFor;
    EndForeach;
    Append(L,Prov);
  EndFor;
  A := L[Len(L)];
  B:= [Elements In A | Elements[Len(Elements)]=2];
  Return [Elements In B | Elements[Len(Elements)-1]< 7];
EndDefine;
```

Remark 3.2. Consider the sequence $\mathbf{H} \ : \ 1 \ \ 3 \ \ \alpha \ \ \beta \ \ \gamma \ \ 2 \ \ 0$. Using the above algorithm for CoCoA, we obtained 79 possible Artinian O-sequences of type 2 (see Example 3.18, [6]). The only 24 cases are all possible level Artinian O-sequences among all 79 possible Artinian O-sequences.

Here we introduce how to construct the most interesting case

$$1 \quad 3 \quad 6 \quad 10 \quad 6 \quad 2$$

among 24 possible level Artinian O-sequences, which is extremal.

$$\mathbb{Z} = \begin{cases} * & \bullet & & \\ * & \bullet & & \\ * & \bullet & \bullet & * \\ \bullet & * & * & * \\ \bullet & * & \bullet & * \\ \bullet & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 20 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 10 & 6 & 2 & 0 & \rightarrow . \end{array}$$

We now consider Question 2.3 for the case $\sigma = 6$. Using the above Georgio's algorithm for CoCoA again, we obtain all possible 62 Artinian O-sequences of type 4 as in Table 1.

Before we find all possible level Artinian O-sequences of socle degree 5 and type 4, we need the following theorems and corollary to eliminate some of non-level cases.

1)	1, 3, 3, 4, 4, 4	2)	1, 3, 3, 4, 5, 4	3)	1, 3, 4, 4, 4, 4
4)	1, 3, 4, 4, 5, 4	5)	1, 3, 4, 5, 4, 4	6)	1, 3, 4, 5, 5, 4
7)	1, 3, 4, 5, 6, 4	8)	1, 3, 5, 4, 4, 4	9)	1, 3, 5, 4, 5, 4
10)	1, 3, 5, 5, 4, 4	11)	1, 3, 5, 5, 5, 4	12)	1, 3, 5, 5, 6, 4
13)	1, 3, 5, 6, 4, 4	14)	1, 3, 5, 6, 5, 4	15)	1, 3, 5, 6, 6, 4
16)	1, 3, 5, 6, 7, 4	17)	1, 3, 5, 7, 4, 4	18)	1, 3, 5, 7, 5, 4
19)	1, 3, 5, 7, 6, 4	20)	1, 3, 5, 7, 7, 4	21)	1, 3, 5, 7, 8, 4
22)	1, 3, 5, 7, 9, 4	23)	1, 3, 6, 4, 4, 4	24)	1, 3, 6, 4, 5, 4
25)	1, 3, 6, 5, 4, 4	26)	1, 3, 6, 5, 5, 4	27)	1, 3, 6, 5, 6, 4
28)	1, 3, 6, 6, 4, 4	29)	1, 3, 6, 6, 5, 4	30)	1, 3, 6, 6, 6, 4
31)	1, 3, 6, 6, 7, 4	32)	1, 3, 6, 7, 4, 4	33)	1, 3, 6, 7, 5, 4
34)	1, 3, 6, 7, 6, 4	35)	1, 3, 6, 7, 7, 4	36)	1, 3, 6, 7, 8, 4
37)	1, 3, 6, 7, 9, 4	38)	1, 3, 6, 8, 4, 4	39)	1, 3, 6, 8, 5, 4
40)	1, 3, 6, 8, 6, 4	41)	1, 3, 6, 8, 7, 4	42)	1, 3, 6, 8, 8, 4
43)	1, 3, 6, 8, 9, 4	44)	1, 3, 6, 8, 10, 4	45)	1, 3, 6, 9, 4, 4
46)	1, 3, 6, 9, 5, 4	47)	1, 3, 6, 9, 6, 4	48)	1, 3, 6, 9, 7, 4
49)	1, 3, 6, 9, 8, 4	50)	1, 3, 6, 9, 9, 4	51)	1, 3, 6, 9, 10, 4
52)	1, 3, 6, 9, 11, 4	53)	1, 3, 6, 9, 12, 4	54)	1, 3, 6, 10, 4, 4
55)	1, 3, 6, 10, 5, 4	56)	1, 3, 6, 10, 6, 4	57)	1, 3, 6, 10, 7, 4
58)	1, 3, 6, 10, 8, 4	59)	1, 3, 6, 10, 9, 4	60)	1, 3, 6, 10, 10, 4
61)	1, 3, 6, 10, 11, 4	62)	1, 3, 6, 10, 12, 4		

TABLE 1. All possible Artinian O-sequences of length 6 and type 4

Theorem 3.3 (Theorem 2.12, [6]). *Let $h_{d-2},\ h_{d-1},\ h_d$ be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d-1\rangle} \qquad \text{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2\rangle}.$$

Let I be any ideal in $R = k[x_1, \dots, x_n]$ such that the Hilbert function of R/I satisfies

$$\begin{aligned} \mathbf{H}(R/I, d-2) &= h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) &= h_{d-1} \\ \mathbf{H}(R/I, d) &= h_d. \end{aligned}$$

Then, the ring R/I has socle of dimension ε in degree $d-2$.

Theorem 3.4 (Theorem 8, [6]). *If* $\mathbf{H}\ :\ h_0\ \ h_1\ \ \cdots\ \ h_t\ \ 0\ \rightarrow$ *is a level Artinian O-sequence where* $t \geq 2$*, then* $\mathbf{G}\ :\ h_0\ \ h_1\ \ \cdots\ \ h_{t-1}\ 0 \rightarrow$ *is also a level Artinian O-sequence.*

Using Theorems 3.3 and 3.4, the following cases in Table 2 are not level.

1)	1, 3, 3, 4, 4, 4	2)	1, 3, 3, 4, 5, 4	4)	1, 3, 4, 4, 5, 4
9)	1, 3, 5, 4, 5, 4	12)	1, 3, 5, 5, 6, 4	24)	1, 3, 6, 4, 5, 4
27)	1, 3, 6, 5, 6, 4	37)	1, 3, 6, 7, 9, 4		

TABLE 2. By Theorems 3.3 and 3.4, the 8 cases are not level.

Theorem 3.5 (Theorem 4.1, [1]). *Let* $R = k[x_1, x_2, x_3]$ *and let* $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ *be the h-vector of a graded Artinian algebra* $A = R/I$ *with socle degree* s*. If*

$$h_{d-1} > h_d \qquad \textit{and} \qquad h_d = h_{d+1} \leq 2d + 3,$$

then $\mathbf{H}$ *is not level*

By Theorem 3.5, the following 12 cases in Table 3 are not level.

5)	1, 3, 4, 5, 4, 4	8)	1, 3, 5, 4, 4, 4	10)	1, 3, 5, 5, 4, 4
13)	1, 3, 5, 6, 4, 4	17)	1, 3, 5, 7, 4, 4	23)	1, 3, 6, 4, 4, 4
25)	1, 3, 6, 5, 4, 4	28)	1, 3, 6, 6, 4, 4	32)	1, 3, 6, 7, 4, 4
38)	1, 3, 6, 8, 4, 4	45)	1, 3, 6, 9, 4, 4	54)	1, 3, 6, 10, 4, 4

TABLE 3

Proposition 3.6. *There is no level Artinian algebra with Hilbert function* $h = (1, 3, 5, 7, 5, 4)$.

```
Define ADDUPHilbert(L)
  S:=[];
    For I := 2 To Len(L)
      Do S1:=Sum(First(L,I));
      Append(S,S1);
    EndFor;
   S2:=Comp(S,Len(S));
   S:=[S];
   Append(S,S2);
   S;
EndDefine;
```

```
ADDUPHilbert([1,3,5,7,5,4]); [[4, 9, 16, 21, 25], 25]
-------------------------------
TV.FromHF([[4,9,16,21,25],25]); [[[1], [2], [3]], [[1], [3], [4],
[5], [6]]]
-------------------------------
TV.FromHF([[4,9,16,21,25],25]); TV.PrintRes(It); [[[1], [2], [3]],
[[1], [3], [4], [5], [6]]];
-------------------------------
```

Proof. Consider the h-vector $h = (1, 3, 5, 7, 5, 4)$ Hilbert function. The minimal free resolution of the lex-segment ideal I^{lex} associated to I is

$$\begin{array}{rl} 0 \;\to & \boldsymbol{R^3}(\boldsymbol{-6}) \oplus R(-7) \oplus R^4(-8) \to R^7(-5) \oplus \boldsymbol{R^3}(\boldsymbol{-6}) \oplus R^8(-7) \\ \to & R(-2) \oplus R^4(-4) \oplus R^2(-5) \oplus R^4(-6) \to R \to R/I^{\text{lex}} \to 0 \end{array}$$

and suppose that $A = R/I, \;\; R = k[x_1, x_2, x_3]$ is a level algebra with h.

Claim : The minimal number of generators of I, in degree 6, is < 4.

Notice that once this claim is proved we are done. Since then $\beta_{0,6}(A) < 4$ and by the Bigatti-Hulett-Pardue result we must have $\beta_{1,6}(A) < 3$. This, in turn, implies that $\beta_{2,6}(A) \geq 1$ and hence A cannot be level.

Proof of Claim. Suppose that I has 4 generators in degree 6 and let $J = (I_{\leq 5})$. Then the Hilbert function of R/J begins $1\ 3\ 5\ 7\ 5\ 4\ 4 \cdots$. Now, we can use Theorem 3.4 in [4] to assert that R/J has 1-dimensional socle in degree 4. Since R/J and R/I agree in degree ≤ 5 we conclude that R/I cannot be level.

□

Using the same idea as in the proof of Proposition, one can show that the following four cases cannot be level;

18)	1, 3, 5, 7, 5, 4	26)	1, 3, 6, 5, 5, 4
33)	1, 3, 6, 7, 5, 4	56)	1, 3, 6, 10, 6, 4

TABLE 4. By Proposition 3.6, the 4 cases are not level.

Example 3.7. Let I_1, I_2, I_3, and I_4 be ideals of $R = k[x_1, x_2, x_3]$ such that Hilbert functions $(1, 3, 6, 6, 7, 4)$, $(1, 3, 6, 8, 5, 4)$, $(1, 3, 6, 9, 5, 4)$, and $(1, 3, 6, 10, 5, 4)$ of $R/I_1, R/I_2, R/I_3$, and R/I_4 are $(1, 3, 6, 6, 7, 4)$, $(1, 3, 6, 8, 5, 4)$, $(1, 3, 6, 9, 5, 4)$ and $(1, 3, 6, 10, 5, 4)$ respectively.

Then the Minimal Free Resolutions of $R/I_1, R/I_2, R/I_3$ and R/I_4 are

$$\begin{aligned} 0 &\rightarrow \boldsymbol{R(-5)} \oplus R^3(-7) \oplus R^4(-8) \rightarrow R^4(-4) \oplus R^7(-6) \oplus R^8(-7) \\ &\rightarrow R^4(-3) \oplus R^4(-5) \oplus R^4(-6) \rightarrow R \rightarrow R/I_1^{\text{lex}} \rightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 &\rightarrow \boldsymbol{R^4(-6)} \oplus R(-7) \oplus R^4(-8) \rightarrow R(-4) \oplus R^9(-5) \oplus \boldsymbol{R^3(-6)} \oplus R^8(-7) \\ &\rightarrow R^2(-3) \oplus R^5(-4) \oplus R^2(-5) \oplus R^4(-6) \rightarrow R \rightarrow R/I_2^{\text{lex}} \rightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 &\rightarrow \boldsymbol{R^5(-6)} \oplus R(-7) \oplus R^4(-8) \rightarrow R^{12}(-5) \oplus \boldsymbol{R^3(-6)} \oplus R^8(-7) \\ &\rightarrow R(-3) \oplus R^7(-4) \oplus R^2(-5) \oplus R^4(-6) \rightarrow R \rightarrow R/I_3^{\text{lex}} \rightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 &\rightarrow \boldsymbol{R^6(-6)} \oplus R(-7) \oplus R^4(-8) \rightarrow R^{15}(-5) \oplus \boldsymbol{R^3(-6)} \oplus R^8(-7) \\ &\rightarrow R^{10}(-4) \oplus R^2(-5) \oplus R^4(-6) \rightarrow R \rightarrow R/I_4^{\text{lex}} \rightarrow 0. \end{aligned}$$

As we see from the above minimal free resolutions of all above four cases last free modules have non-cancelable shifts, and hence they cannot be level.

4. Some Construction of Gorenstein and Level Artinian O-sequences

Definition 4.1 (Definition 3.1, [11])**.** (a) A finite set $\mathbb{X}$ of points $\mathbb{P}^2$ is called a ***basic configuration*** of type (d, e) if there exist distinct elements b_j, c_j in k such that

$$I_{\mathbb{X}} = \left(\prod_{j=1}^{d} (x - b_j z), \prod_{j=1}^{e} (y - c_j z) \right).$$

We denote $\mathbb{X} := \mathbb{B}(d, e)$.

(b) A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a ***pure configuration*** if there exist finite basic configurations $\mathbb{B}(d_1, e_1), \ldots, \mathbb{B}(d_m, e_m)$ where $e_1 > \cdots > e_m$, which satisfy the following three conditions:

(i) $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \varnothing$ if $i \neq j$,

(ii) $\mathbb{X} = \mathbb{B}(d_1, e_1) \bigcup \cdots \bigcup \mathbb{B}(d_m, e_m)$,

(iii) $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$ for all $1 \leq i \leq m-1$, where $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \to \mathbb{P}^1$ is the map defined by sending the point (x, y, z) to the point (y, z). In this case, we denote $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$.

Proposition 4.2 (Proposition 3.8, [10]). *Let* $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ *be a pure configuration in* $\mathbb{P}^2$. *Then a minimal free resolution of* $\mathbb{X}$ *is :*

$$0 \to \bigoplus_{i=1}^{m} R(-p_i) \to \bigoplus_{i=1}^{m+1} R(-q_i) \to R \to R/I_{\mathbb{X}} \to 0,$$

where

$$q_1 = e_1, \quad q_i = d_1 + \cdots + d_{i-1} + e_i \quad (2 \leqslant i \leqslant m),$$

$$q_{m+1} = d_1 + \cdots + d_m, \quad p_i = q_i + d_i \quad (1 \leqslant i \leqslant m).$$

Corollary 4.3 (Corollary 3.10, [10]). *Let* $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ *be a pure configuration in* $\mathbb{P}^2$. *Then* $\mathbb{X}$ *is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all $1 \leqslant i \leqslant m - 1$.

If $r(A)$ is Cohen-Macaulay type of a Cohen-Macaulay standard graded k-algebra A, and if $A = \bigoplus_{i \geq 0} A_i$ is an Artinian level algebra, then $r(A) = \dim_k A_{\sigma(A)-1}$, where $\sigma(A) = \min\{i | A_i = 0\}$. If $\mathbb{Z} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ is a level pure configuration in $\mathbb{P}^2$, then $r(\mathbb{Z}) = r(R/I_{\mathbb{Z}}) = m$.

Lemma 4.4 (Lemma 3.14, [10]). *Let* $\mathbb{Z}$ *be a level set of points in* $\mathbb{P}^n$ *and* $\mathbb{X}$ *a subset of* $\mathbb{Z}$. *Set* $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. *Then* $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ *is an Artinion level graded* k*-algebra with* $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \sigma(\mathbb{Z}) - 1$ *and* $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq r(\mathbb{Z})$.

Corollary 4.5 (Corollary 3.15, [10]). *Let $\mathbb{Z} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ be a level pure configuration in $\mathbb{P}^2$ and $\mathbb{X}$ a subset of $\mathbb{Z}$. Set $\mathbb{Y} := \mathbb{Z} \setminus \mathbb{X}$. Then $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is an Artinion level graded k-algebra with $\sigma(R/(I_{\mathbb{X}}+I_{\mathbb{Y}})) = d_1 + e_1 - 2$ and $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq m$.*

Now we classify all cases in Table 1 in the following example.

Example 4.6. Now we construct 29 level Artinian O-sequences among 62 cases in Table 1, which are: 3), 6), 7), 11), 14), 15), 16), 19), 20), 21), 22), 29), 30), 34), 35), 36), 40), 41), 42), 43), 44), 48), 49), 50), 51), 57), 58), 59), 60).

3)

$$\mathbb{Z} = \left\{ \begin{array}{ccccccc} * & \bullet & & & & & \\ * & * & & & & & \\ * & * & * & * & & & \\ \bullet & * & * & * & & & \\ * & * & \bullet & * & * & * & \\ * & * & * & * & * & * & \bullet \end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 4 & 4 & 4 & 4 & 4 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 21 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 4 & 4 & 4 & 4 & 0 & \rightarrow . \end{array}$$

6)

$$\mathbb{Z} = \left\{\begin{array}{ccccccc} \ast & \ast & & & & & \\ \ast & \ast & & & & & \\ \ast & \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \ast & \bullet & \\ \bullet & \bullet & \bullet & \ast & \bullet & \ast & \ast \end{array}\right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 4 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 20 & 20 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 4 & 5 & 5 & 4 & 0 & \rightarrow . \end{array}$$

7)

$$\mathbb{Z} = \left\{\begin{array}{ccccccc} \ast & \ast & & & & & \\ \ast & \bullet & & & & & \\ \ast & \bullet & \ast & \ast & & & \\ \ast & \bullet & \ast & \ast & & & \\ \ast & \bullet & \ast & \bullet & \ast & \ast & \\ \ast & \bullet & \ast & \ast & \ast & \ast & \ast \end{array}\right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 4 & 5 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 19 & 19 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 4 & 5 & 6 & 4 & 0 & \rightarrow . \end{array}$$

11)

$$\mathbb{Z} = \left\{\begin{array}{ccccccc} \ast & \ast & & & & & \\ \ast & \ast & & & & & \\ \ast & \ast & \ast & \ast & & & \\ \ast & \ast & \bullet & \ast & & & \\ \bullet & \ast & \ast & \ast & \ast & \bullet & \\ \bullet & \ast & \ast & \ast & \ast & \ast & \bullet \end{array}\right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 20 & 20 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 5 & 5 & 4 & 0 & \rightarrow . \end{array}$$

14)

$$\mathbb{Z} = \begin{cases} * \\ * & * \\ * & * & \bullet \\ * & * & * & * \\ * & * & \bullet & * \\ \bullet & * & \bullet & * \\ * & \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 13 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 6 & 5 & 4 & 0 & \rightarrow . \end{array}$$

15)

$$\mathbb{Z} = \begin{cases} * & * \\ \bullet & * \\ * & \bullet & * & * \\ * & \bullet & \bullet & * \\ * & * & * & \bullet & * & * \\ * & \bullet & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 19 & 19 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 6 & 6 & 4 & 0 & \rightarrow . \end{array}$$

16)

$$\mathbb{Z} = \begin{cases} * & * \\ * & \bullet \\ * & * & \bullet & * \\ * & * & * & \bullet \\ * & * & * & \bullet & \bullet & * \\ \bullet & * & * & * & * & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 6 & 7 & 7 & 7 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 18 & 18 & \rightarrow & \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 6 & 7 & 4 & 0 & \rightarrow & . \end{array}$$

19)

$$\mathbb{Z} = \begin{cases} \ast & & & \\ \ast & \ast & & \\ \bullet & \ast & \bullet & \\ \bullet & \ast & \bullet & \ast \\ \bullet & \ast & \bullet & \ast \\ \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \ast \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 7 & 7 & 7 & 7 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 13 & 15 & 15 & \rightarrow & \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 7 & 6 & 4 & 0 & \rightarrow & . \end{array}$$

20)

$$\mathbb{Z} = \begin{cases} \ast & \ast & & & & & \\ \ast & \ast & & & & & \\ \bullet & \ast & \ast & \ast & & & \\ \ast & \ast & \bullet & \ast & & & \\ \ast & \ast & \ast & \ast & \bullet & \ast & \\ \ast & \bullet & \bullet & \bullet & \ast & \bullet & \ast \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 7 & 7 & 7 & 7 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 18 & 18 & \rightarrow & \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 7 & 7 & 4 & 0 & \rightarrow & . \end{array}$$

21)

$$\mathbb{Z} = \begin{cases} * & * & & & & & \\ * & * & & & & & \\ * & * & * & * & & & \\ * & * & * & * & & & \\ \bullet & \bullet & * & \bullet & \bullet & * & \\ * & \bullet & * & \bullet & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 7 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 17 & 17 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 7 & 8 & 4 & 0 & \rightarrow . \end{array}$$

22)

$$\mathbb{Z} = \begin{cases} \bullet & * & & & & & \\ \bullet & * & & & & & \\ * & \bullet & \bullet & * & & & \\ * & * & \bullet & * & & & \\ * & * & * & \bullet & \bullet & * & \\ * & * & * & * & \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 7 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 7 & 9 & 4 & 0 & \rightarrow . \end{array}$$

29)

$$\mathbb{Z} = \begin{cases} * & & & \\ * & * & & \\ * & * & * & \\ * & * & * & \bullet \\ * & \bullet & \bullet & * \\ * & * & * & \bullet \\ \bullet & * & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 13 & 16 & 16 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 6 & 5 & 4 & 0 & \rightarrow .
\end{array}$$

30)

$$\mathbb{Z} = \begin{cases}
\ast & \ast \\
\ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \bullet & \ast \\
\ast & \bullet & \ast & \ast & \bullet & \bullet \\
\ast & \ast & \ast & \ast & \bullet & \bullet & \ast
\end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 19 & 19 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 6 & 6 & 4 & 0 & \rightarrow .
\end{array}$$

34)

$$\mathbb{Z} = \begin{cases}
\ast \\
\ast & \bullet \\
\ast & \bullet & \bullet \\
\ast & \ast & \ast & \ast \\
\bullet & \ast & \bullet & \ast \\
\ast & \ast & \bullet & \ast \\
\ast & \ast & \bullet & \ast
\end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 13 & 15 & 15 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 7 & 6 & 4 & 0 & \rightarrow .
\end{array}$$

35)

$$\mathbb{Z} = \begin{cases} * & * \\ \bullet & * \\ \bullet & * & * & * \\ * & \bullet & \bullet & \bullet \\ * & * & * & * & \bullet & * \\ * & * & * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 18 & 18 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 7 & 7 & 4 & 0 & \rightarrow . \end{array}$$

36)

$$\mathbb{Z} = \begin{cases} * & \bullet \\ * & * \\ * & * & * & * \\ * & \bullet & * & * \\ * & \bullet & * & * & * & * \\ \bullet & * & \bullet & \bullet & \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 7 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 17 & 17 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 7 & 8 & 4 & 0 & \rightarrow . \end{array}$$

40)

$$\mathbb{Z} = \begin{cases} \bullet \\ * & * \\ * & * & * \\ * & * & * & \bullet \\ * & * & \bullet & * \\ * & * & \bullet & \bullet \\ \bullet & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rccccccccc}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 8 & 8 & 8 & 8 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 12 & 14 & 14 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 8 & 6 & 4 & 0 & \rightarrow .
\end{array}$$

41)

$$\mathbb{Z} = \begin{cases}
\ast & & & \\
\bullet & \ast & & \\
\bullet & \ast & \bullet & \\
\bullet & \bullet & \ast & \ast \\
\bullet & \ast & \ast & \ast \\
\bullet & \bullet & \ast & \ast \\
\bullet & \ast & \bullet & \ast
\end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rccccccccc}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 8 & 9 & 10 & 10 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 12 & 12 & 12 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 8 & 7 & 4 & 0 & \rightarrow .
\end{array}$$

42)

$$\mathbb{Z} = \begin{cases}
\ast & \ast & & & & & \\
\ast & \ast & & & & & \\
\ast & \ast & \bullet & \ast & & & \\
\ast & \bullet & \bullet & \ast & & & \\
\ast & \ast & \ast & \bullet & \ast & \ast & \\
\ast & \bullet & \bullet & \ast & \bullet & \ast & \bullet
\end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rccccccccc}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 8 & 8 & 8 & 8 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 17 & 17 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 8 & 8 & 4 & 0 & \rightarrow .
\end{array}$$

43)

$$\mathbb{Z} = \left\{ \begin{array}{lllllll} \bullet & \ast & & & & & \\ \bullet & \ast & & & & & \\ \bullet & \ast & \bullet & \ast & & & \\ \ast & \ast & \bullet & \ast & & & \\ \bullet & \ast & \ast & \ast & \ast & \bullet & \\ \bullet & \ast & \bullet & \ast & \ast & \ast & \ast \end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rclllllll} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 8 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 8 & 9 & 4 & 0 & \rightarrow . \end{array}$$

44)

$$\mathbb{Z} = \left\{ \begin{array}{lllllll} \ast & \ast & & & & & \\ \ast & \ast & & & & & \\ \ast & \ast & \ast & \bullet & & & \\ \ast & \ast & \ast & \ast & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ast & \\ \bullet & \bullet & \bullet & \ast & \ast & \bullet & \ast \end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rclllllll} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 8 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 8 & 10 & 4 & 0 & \rightarrow . \end{array}$$

48)

$$\mathbb{Z} = \left\{ \begin{array}{llll} \ast & & & \\ \bullet & \ast & & \\ \ast & \bullet & \ast & \\ \ast & \bullet & \bullet & \ast \\ \ast & \ast & \bullet & \ast \\ \ast & \bullet & \bullet & \bullet \\ \ast & \bullet & \bullet & \bullet \end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 9 & 10 & 11 & 11 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 9 & 7 & 4 & 0 & \rightarrow . \end{array}$$

49)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ \bullet & * \\ * & * & * & \bullet \\ \bullet & * & * & * \\ * & * & \bullet & \bullet & \bullet & \bullet \\ * & * & * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 14 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 9 & 8 & 4 & 0 & \rightarrow . \end{array}$$

50)

$$\mathbb{Z} = \begin{cases} * & \bullet \\ * & * \\ \bullet & * & * & * \\ * & * & \bullet & \bullet \\ \bullet & \bullet & \bullet & * & * & * \\ * & \bullet & * & * & \bullet & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 9 & 9 & 4 & 0 & \rightarrow . \end{array}$$

51)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ * & * \\ * & * & * & * \\ * & * & * & \bullet \\ \bullet & \bullet & \bullet & \bullet & * & \bullet \\ \bullet & \bullet & * & * & \bullet & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 9 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 15 & 15 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 9 & 10 & 4 & 0 & \rightarrow .
\end{array}$$

57)

$$\mathbb{Z} = \begin{cases}
\bullet & & & \\
* & \bullet & & \\
* & * & * & \\
* & \bullet & \bullet & \bullet \\
* & \bullet & * & * \\
* & * & * & \bullet \\
* & \bullet & \bullet & \bullet
\end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 11 & 12 & 12 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 10 & 7 & 4 & 0 & \rightarrow .
\end{array}$$

58)

$$\mathbb{Z} = \begin{cases}
\bullet & * & & & & & \\
\bullet & * & & & & & \\
* & * & \bullet & * & & & \\
\bullet & \bullet & \bullet & * & & & \\
\bullet & * & \bullet & \bullet & * & * & \\
* & * & * & * & \bullet & * & *
\end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 13 & 15 & 15 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 10 & 8 & 4 & 0 & \rightarrow .
\end{array}$$

59)

$$\mathbb{Z} = \left\{ \begin{array}{lllllll} \bullet & \ast & & & & & \\ \bullet & \ast & & & & & \\ \bullet & \ast & \ast & \ast & & & \\ \ast & \ast & \bullet & \ast & & & \\ \ast & \ast & \bullet & \bullet & \bullet & \ast & \\ \ast & \ast & \ast & \ast & \bullet & \bullet & \bullet \end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 14 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 10 & 9 & 4 & 0 & \rightarrow . \end{array}$$

60)

$$\mathbb{Z} = \left\{ \begin{array}{lllllll} \ast & \ast & & & & & \\ \bullet & \bullet & & & & & \\ \ast & \ast & \ast & \ast & & & \\ \ast & \bullet & \bullet & \bullet & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast & \\ \bullet & \ast & \ast & \bullet & \bullet & \ast & \bullet \end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 10 & 10 & 4 & 0 & \rightarrow . \end{array}$$

Theorem 4.7 (Proposition 5.24, [6]). *Let* $(1, h_1, h_2, \cdots, h_s)$ *be the h-vector of a level algebra* $A = R/I$ *where* $R = k[x, y, z]$. *Let* $j = \max\{i \mid h_i = \binom{2+i}{i}\}$. *Then, there is an ideal* $J \subset I$ *so that* $B = R/J$ *is a level algebra with h-vector* $(1, h_1, \ldots, h_j, h_{j+1} + 1, \ldots, h_s + 1)$.

To use Theorem 4.7, we now construct the level O-sequence $(1, 3, 6, 7, 4, 2)$ using link-sum construction.

Example 4.8. We start with a level set of points, $\mathbb{Z} = \mathbb{B}(7,1) \cup \mathbb{B}(4,3)$ in $\mathbb{P}^2$ which we partition into two subsets $\mathbb{X}$ and $\mathbb{Y}$. The points of $\mathbb{X}$ will be denoted with $\bullet$'s and those of $\mathbb{Y}$ with $*$'s. Next to each diagram we give the Hilbert functions of all the relevant rings.

$$\mathbb{Z} = \begin{cases} * & & & \\ * & & & \\ * & & & \\ \bullet & * & * & * \\ * & * & \bullet & \bullet \\ * & \bullet & \bullet & * \\ * & \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 17 & 19 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 11 & 12 & 12 & \rightarrow & \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 7 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

Now we apply Theorem 4.7 to get the level Artinian O-sequence $(1, 3, 6, 7+1, 4+1, 2+1) = (1, 3, 6, 8, 5, 3)$. And applying Theorem 4.7 to a level Artinian O-sequence again, we obtain $(1, 3, 6, 8+1, 5+1, 3+1) = (1, 3, 6, 9, 6, 4)$.

Proposition 4.9 (Remark 5.32, [6])**.** *The link-sum construction cannot be used to construct level Artinian algebras of type* 4 *in codimension* 3 *with Hilbert function* $(1, 3, \alpha, \beta, \gamma, 4)$ *and* $\gamma = 11, 12$ *from level sets of points in* $\mathbb{P}^2$*.*

Proof. We shall show that $(1,3,\alpha,\beta,s,4)$ cannot be constructed for $s=11,12$. Using level sets of points in $\mathbb{P}^2$. Suppose we had a level set of points $\mathbb{Z}$ in $\mathbb{P}^2$ and $\mathbb{Z}=\mathbb{X}\dot{\cup}\mathbb{Y}$ which gave Hilbert Function as above by the link-sum construction. Then if we let $A=R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$, we would have a display

$$\begin{array}{rccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & - & - & b & c & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & - & - & \alpha_1 & \alpha_2 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & - & - & \beta_1 & \beta_2 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & - & - & s & 4 & 0 & . \end{array}$$

where $11 \leq s \leq 12$.

The construction method says that $b+s=\alpha_1+\beta_1$ and $c+4=\alpha_2+\beta_2$. By subtracting, we have

$$(c-b)+(4-s)=(\alpha_2-\alpha_1)+(\beta_2-\beta_1).$$

But, since $\mathbb{X}$ and $\mathbb{Y}$ are point sets in $\mathbb{P}^2$ we have $(\alpha_2-\alpha_1)\geq 0$ and $(\beta_2-\beta_1)\geq 0$. Since $4-s\leq -7$ we must have $c-b\geq 7$. But, since $\mathbb{Z}$ has a differentiable Hibert function, the maximum value for $c-b$ is $6=\dim_k k[x,y]_5$, which is a contradiction, as we wished. □

By Proposition 4.9, the following cases in Table cannot be obtained using the link-sum construction.

52)	1, 3, 6, 9, 11, 4	61)	1, 3, 6, 10, 11, 4
53)	1, 3, 6, 9, 12, 4	62)	1, 3, 6, 10, 12, 4

TABLE 5

However, in [6], they showed that the above O-sequence in Table 5 are all level (see Example 6.18 [6]).

References

[1] J.M. Ahn and Y.S. Shin, *Generic Initial Ideals and Graded Artinian Leval Algebras not having the Weak-Lefschetz Property.* **210** (2007) 855-879.

[2] A.M. Bigatti, *Upper Bounds for the Betti Numbers of a Given Hilbert Function.* Comm. in Alg. **21**(7) (1993) 2317-2334.

[3] A. Capani, G. Niesi and L. Robbiano, *CoCoA, a system for doing Computations in Commutative Algebra.* Available via anonymous ftp from: `cocoa.dima.unige.it`.

[4] F. Fröberg and D. Laksov, *Compressed Algebras, Complete Intersections (Acireale, 1983).* Lecture Notes in Math. Springer-Verlag, **1092** (1984) 121–151.

[5] A.V. Geramita, Waring's Problem for Forms: inverse systems of fat points, secant varieties and Gorenstein algebras Queen's Papers in Pure and Applied Math. The Curves Seminar, Vol. X.**105** (1996)

[6] A.V. Geramita, T. Harima, J.C. Migliore and Y.S. Shin, *The Hilbert Function of a Level Algebra Memoris of the American Mathematical Society,.* Vol.180, (2007) No.872, 1∼139.

[7] A.V. Geramita, T. Harima and Y.S. Shin, *An Alternative to the Hilbert function for the ideal of a finite set of points in* $\mathbb{P}^n$. Illinois J. of Math. **45**(1) (2001) 1–23.

[8] A.V. Geramita, T. Harima and Y.S. Shin, *Extremal point sets and Gorenstein ideals.* Adv. in Math. **152** (2000) 78–119.

[9] A.V. Geramita, T. Harima and Y.S. Shin, *Decompositions of the Hilbert function of a set of points in* $\mathbb{P}^n$. Can. J. of Math. **53**(5) (2001) 923–943.

[10] A.V. Geramita, T. Harima and Y.S. Shin, *Some Special Configurations of Points in* $\mathbb{P}^n$. J. of Alg. To appear (2002).

[11] T. Harima, *A Note on Artinian Gorenstein Algebras of Codimension Three.* J. of Pure and App. Alg. **135** (1999) 45–56.

[12] H.A. Hulett, *Maximum Betti Numbers of Homogeneous Ideals with a Given Hilbert Function.* Comm. in Alg. **21**(7) (1993) 2335–2350.

[13] K. Pardue, *Deformation Classes of Graded Modules and Maximal Betti Numbers.* Illinois J. of Math. **40** (1996) 564–585.

ABSTRACT

Some Construction of All Level Artinian O-sequences of Socle Degree 5 and Type 4

Se-ryoung, Jung
Major in Mathematics Education
Graduate school of Education
Sungshin Women's University
Supervised by Professor Shin Yong su Ph.D.

The purpose of this study was to examine the Codimension 3, Socle Degree 5 and Type 4 Artinian O-sequences 62 based on Theorem of Fröberg and Laksov.

First of all, we prove that 28 sequences cannot be level using [1], [4], [6].

Then we introduce how to construct 29 sequences using CoCoA, S-plus. And we prove that 5 sequences can be level by Theorem in [6].