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석사학위 청구논문

Some Construction of All Level Artinian O-sequences of Socle Degree 5 and Type 2

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이 논문을 석사학위논문으로 제출함.

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이 경 애

인 준 서

이경애의 석사학위 논문으로 인준함.

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논 문 개 요

Fröberg and Laksov 의 정리에 근거하여 여차원이 3, 형태가 2, 길이가 6인 level이 될 가능성이 있는 79개의 Artinian O-sequences에 대하여 level 여부를 연구하였다.

우선 참고문헌 Generic Initials and Graded Artinian Level Algebras not having the Weak-Lefschetz Property, The Hilbert Function of a Level Algebra Memoris of the American Mathemetical Society, Smooth points in Gor(T)을 참고하여 level이 될 가능성이 없는 53개를 제외하였다.

23개의 level이 되는 수열은 Reference의 참고문헌과 CoCoA, S-plus를 이용하여 “link-sum”을 구성하였다.

CONTENTS

1. **Introduction**

Let k be an infinite field, $R = k[x_1, \ldots, x_n] = \bigoplus_{i \geq 0} R_i$, and let I be a homogeneous ideal of R, $A = R/I$. The ***Hilbert function of*** A, $\mathbf{H}_A : \mathbb{N} \longrightarrow \mathbb{N}$, defined by,

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t$$

has been much studied and, in case I is the ideal of a subscheme $\mathbb{X}$ of $\mathbb{P}^{n-1}$, This function contains a great deal of information about the geometry of this subscheme.

Let $\mathbf{H}$ be a numerical function which can be the Hilbert function of a finite set of points in $\mathbb{P}^n$. For a given $\mathbf{H}$, there are infinitely many different collections of points which share $\mathbf{H}$ as their Hilbert function. In [7], they showed that there is a one to one correspondence between all possible Hilbert functions of finite sets of points in $\mathbb{P}^n$ and n-type vectors (see Theorem 2.6 [7], see also [8] or [9]).

In [6], they asked the following questions.

(1) If $\mathbf{H}$ is a possible Hilbert function for a set of reduced points in $\mathbb{P}^n$, what are all the possible graded Betti numbers for sets of points $\mathbb{X} \subseteq \mathbb{P}^n$ which have Hilbert function $\mathbf{H}$?

(2) What can be the Hilbert function of a level Artinian algebra, (see Section 2 for the definition)?

By works of Bigatti [2], Hulett [13], and Pardue [14], we know that for a given $\mathbf{H}$ there is a unique maximal set of graded Betti numbers which can have.

In Section 2, we introduce some definitions and preliminary observations about Gorenstein and level Artinian algebras (see also [10]). In [10], they used the fact that if a set of points in $\mathbb{P}^2$ has a level coordinate ring, then one can break the set into two subsets and use those subsets to construct a level algebra of codimension 3. Moreover, they used this method to get a complete list (in codimension 3) of all Hilbert functions of level algebras of type 2 and socle degree ≤ 5. Recently, in [6], they obtained all possible Hilbert functions (in codimension 3) of level algebras of socle degree 6 and type 2.

In Section 3, we prove that same cases cannot be level using several known results Corollary 2.11 in [6] (see Propositions 3.5, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14) and Corollary 3.5 in [6] (see Table 2). We also use Proposition 3.9 in [6] to prove non-level cases in Table 5.

In Section 4, we prove that some sepecial O-sequence cannot be level using the inverse system (or Macaulay duality) (see Propostions 4.6)

In Section 5, we introduce how to construct all codimension 3 level Artinian algebras of socle degree 5 and type 2 using Link-Sum method. Computer programs CoCoA [3] and Splus are used to produce all possible level O-sequences socle degree 5 and type 2 and examples in this thesis.

2. **Preliminary Definitions and Results**

Definition-Proposition 2.1 (Definition-Proposition 2.21 [10]). Let $R = k[x_0, \cdots \ldots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. Then

(a) If $B = B_0 \bigoplus \cdots \bigoplus B_\ell$ ($B_\ell \neq 0$) is an Artinian algebra, then B is ***level*** if and only if $B_\ell = \mathrm{Ann}(B_1)$.

(b) A is a ***level algebra*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. $\operatorname{rank} \mathcal{F}_{n-(d-1)} =$ Cohen-Macaulay type of A.

(c) i) If $\mathbb{X}$ is a non-degenerate set of points in $\mathbb{P}^n$, $A = R/I_{\mathbb{X}}$ its coordinate ring, then we say that ℓ is the ***socle degree*** of $\mathbb{X}$ if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where $\overline{L}$ is any linear non-zero-divisor of A.

ii) $\mathbb{X}$ is called a ***level set*** of points if $A = R/I_{\mathbb{X}}$ is a level algebra. In this case, the socle degree of $\mathbb{X}$ is $\ell = \sigma(\mathbb{X}) + n - 1$.

(d) If $\overline{L}$ is a linear non-zero divisor in $A = R/I_{\mathbb{X}}$, then A is level if and only if $A/\overline{L}A \simeq A/(L, I_{\mathbb{X}})$ is level.

(e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an

Artinian algebra, see e.g., [9]) $b = \{b_i\}_{i\geq 0}$ with $b_1 = n+1$, is called ***level*** if there is a level set of points in $\mathbb{P}^n$ with Hilbert function b.

Here we have an interesting question.

Question 2.2. What are the possible Hilbert functions of level Artinian algebras of codimension 3 with $r(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}))=2$?

We recall a special case of a theorem of Fröberg and Laksov of [4] that if A is a level Artinian algebra of codimension 3 and type m with $\sigma - 1 = t$, then

$$\mathbf{H}(A,i) \leq \min\left\{\binom{2+i}{2}, m\binom{t-i+2}{2}\right\}.$$

The following algorithm for CoCoA is due to Georgio Daldozzo in Italy, and we use this algorithm to obtain all possible Artinian O-sequences of socle degree 5 and type 2 based on Fröberg and Laksov result (see Table 1).

Algorithm 2.3. `Define HvectorAdmis2(K)`

```
L := [[[1,3]]];
For I:=2 To K Do
  Prov := [];
  Foreach El In L[I-1] Do
```

```
      For J:=1 To BinExp(El[Len(El)],I-1,1,1) Do
    Append(Prov,Concat(El,[J]));
      EndFor;
    EndForeach;
    Append(L,Prov);
  EndFor;
  A := L[Len(L)];
  B:= [Elements In A | Elements[Len(Elements)]=2];
  Return [Elements In B | Elements[Len(Elements)-1]< 7];
EndDefine;
```

We now consider Question 2.2 for the case $\sigma = 6$. Using the above Georgio's algorithm for CoCoA again, we obtain all possible 79 Artinian O-sequences of type 2 as in Table 1.

3. **Non existence of Level O-sequences of Scole Degree 5 and Type 2**

Before we find all possible level Artinian O-sequences of socle degree 5 and type 2, we need the following theorems and corollary to eliminate some of non-level cases among all 79 cases in Table 1.

1)	1, 3, 2, 2, 2, 2	2)	1, 3, 3, 2, 2, 2	3)	1, 3, 3, 3, 2, 2
4)	1, 3, 3, 3, 3, 2	5)	1, 3, 3, 4, 2, 2	6)	1, 3, 3, 4, 3, 2
7)	1, 3, 3, 4, 4, 2	8)	1, 3, 3, 4, 5, 2	9)	1, 3, 4, 2, 2, 2
10)	1, 3, 4, 3, 2, 2	11)	1, 3, 4, 3, 3, 2	12)	1, 3, 4, 4, 2, 2
13)	1, 3, 4, 4, 3, 2	14)	1, 3, 4, 4, 4, 2	15)	1, 3, 4, 4, 5, 2
16)	1, 3, 4, 5, 2, 2	17)	1, 3, 4, 5, 3, 2	18)	1, 3, 4, 5, 4, 2
19)	1, 3, 4, 5, 5, 2	20)	1, 3, 4, 5, 6, 2	21)	1, 3, 5, 2, 2, 2
22)	1, 3, 5, 3, 2, 2	23)	1, 3, 5, 3, 3, 2	24)	1, 3, 5, 4, 2, 2
25)	1, 3, 5, 4, 3, 2	26)	1, 3, 5, 4, 4, 2	27)	1, 3, 5, 4, 5, 2
28)	1, 3, 5, 5, 2, 2	29)	1, 3, 5, 5, 3, 2	30)	1, 3, 5, 5, 4, 2
31)	1, 3, 5, 5, 5, 2	32)	1, 3, 5, 5, 6, 2	33)	1, 3, 5, 6, 2, 2
34)	1, 3, 5, 6, 3, 2	35)	1, 3, 5, 6, 4, 2	36)	1, 3, 5, 6, 5, 2
37)	1, 3, 5, 6, 6, 2	38)	1, 3, 5, 7, 2, 2	39)	1, 3, 5, 7, 3, 2
40)	1, 3, 5, 7, 4, 2	41)	1, 3, 5, 7, 5, 2	42)	1, 3, 5, 7, 6, 2
43)	1, 3, 6, 2, 2, 2	44)	1, 3, 6, 3, 2, 2	45)	1, 3, 6, 3, 3, 2
46)	1, 3, 6, 4, 2, 2	47)	1, 3, 6, 4, 3, 2	48)	1, 3, 6, 4, 4, 2
49)	1, 3, 6, 4, 5, 2	50)	1, 3, 6, 5, 2, 2	51)	1, 3, 6, 5, 3, 2
52)	1, 3, 6, 5, 4, 2	53)	1, 3, 6, 5, 5, 2	54)	1, 3, 6, 5, 6, 2
55)	1, 3, 6, 6, 2, 2	56)	1, 3, 6, 6, 3, 2	57)	1, 3, 6, 6, 4, 2
58)	1, 3, 6, 6, 5, 2	59)	1, 3, 6, 6, 6, 2	60)	1, 3, 6, 7, 2, 2
61)	1, 3, 6, 7, 3, 2	62)	1, 3, 6, 7, 4, 2	63)	1, 3, 6, 7, 5, 2
64)	1, 3, 6, 7, 6, 2	65)	1, 3, 6, 8, 2, 2	66)	1, 3, 6, 8, 3, 2
67)	1, 3, 6, 8, 4, 2	68)	1, 3, 6, 8, 5, 2	69)	1, 3, 6, 8, 6, 2
70)	1, 3, 6, 9, 2, 2	71)	1, 3, 6, 9, 3, 2	72)	1, 3, 6, 9, 4, 2
73)	1, 3, 6, 9, 5, 2	74)	1, 3, 6, 9, 6, 2	75)	1, 3, 6, 10, 2, 2
76)	1, 3, 6, 10, 3, 2	77)	1, 3, 6, 10, 4, 2	78)	1, 3, 6, 10, 5, 2
79)	1, 3, 6, 10, 6, 2				

Table 1. All possible Artinian O-sequences of length 6 and type 2

Theorem 3.1 (Theorem 2.12 [6]). *Let $h_{d-2},\ h_{d-1},\ h_d$ be three non-zero positive integers such that*

$$h_d = h_{d-1}^{\langle d-1\rangle} \qquad \text{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2\rangle}.$$

Let I be a homogeneous ideal in $R = k[x_1, \ldots, x_n]$ such that the Hilbert function of R/I satisfies

$$\begin{array}{rcl} \mathbf{H}(R/I, d-2) & = & h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) & = & h_{d-1} \\ \mathbf{H}(R/I, d) & = & h_d. \end{array}$$

Then the ring R/I has socle of dimension ε in degree $d-2$.

Corollary 3.2 (Corollary 3.5, [6])**.** *Let $h = (1, h_1, \ldots, h_t)$ be the h-vector of an Artinian quotient of R with socle degree t. Suppose that for some $d-2 < t$. We have three integers $\acute{h}_{d-2}, \acute{h}_{d-1}$ and h_d satisfying :*

$$h_d = h_{d-1}^{\langle d-1 \rangle} \qquad \textit{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2 \rangle}.$$

If

$$h_{d-2} = h_{d-2} + \varepsilon > 0, \quad h_{d-1} = h_{d-1}, \qquad \textit{and} \qquad h_d = h_d$$

then h is not a level O-sequence.

Using Theorem 3.1 and the Corollary 3.2, the following 14 cases in Table 2 are not level.

Theorem 3.3 (Theorem 8 [6])**.** *If* $\mathbf{H} \ : \ h_0 \ \ h_1 \ \ \cdots \ \ h_t \ \ 0$ *is a level Artinian O-sequence where $t \geq 2$, then* $\mathbf{G} \ : \ h_0 \ \ h_1 \ \ \cdots \ \ h_{t-1} \ 0$ *is also a level Artinian O-sequence.*

6)	1, 3, 3, 4, 3, 2	7)	1, 3, 3, 4, 4, 2	8)	1, 3, 3, 4, 5, 2
15)	1, 3, 4, 4, 5, 2	27)	1, 3, 5, 4, 5, 2	32)	1, 3, 5, 5, 6, 2
39)	1, 3, 5, 7, 3, 2	49)	1, 3, 6, 4, 5, 2	54)	1, 3, 6, 5, 6, 2
56)	1, 3, 6, 7, 3, 2	61)	1, 3, 6, 7, 3, 2	66)	1, 3, 6, 8, 3, 2
71)	1, 3, 6, 9, 3, 2	76)	1, 3, 6, 10, 3, 2		

TABLE 2

Theorem 3.4 (Corollary 2.11 [6]). *Let $h = (1, h_1, \dots, h_s)$ be a level sequence with $h_s \geq 1$, and let A be any Artinian algebra with $\mathbf{H}_A = h$. Then the vector $(h_s, h_{s-1}, \dots, h_1, 1)$ is the sum of:*

1) *a Gorenstein sequence $(1, b_1, \dots, b_{s-1}, 1)$ (of length $s+1$) which is the h-vector of a quotient of A and*
2) *$h_s - 1$ O-sequences $(1, d_{j1}, \dots, d_{jr_j})$, $j = 1, \dots h_s - 1$, each of which is the h-vector of a quotient of A.*

Proposition 3.5. *An Artinian algebra with Hilbert function, $h = (1, 3, \alpha, \dots, 5, 3, 2)$ is not level.*

Proof. Assume there is an Artinian algebra A with Hilbert function h. By Theorem 3.3 the only possibilities for α are $3, 4, 5$ and 6. Then, by Theorem 3.4, the vector $(2, 3, 5, \dots, \alpha, 3, 1)$ is the sum of a Gorenstein sequence $(1, b_1, \dots, b_{s-1}, 1)$ of length $s+1$ and one of the possible h_{s-1} O-sequences $(1, b_1, \dots)$. and we have the details as follows.

Case 1.

Gorenstein sequence	:	1,	1,	1,	...	1,	1,	**1**
O-sequence	:	1,	2,	4,	...	$*$,	2,	**1**,

which is impossible.

Case 2.

Gorenstein sequence	:	1,	2,	2,	...	2,	2,	1
O-sequence	:	1,	**1**,	**3**,	...	$*$,	$*$,	$*$,

which is impossible.

Case 3.

Gorenstein sequence	:	1,	2,	3,	...	3,	2,	1
O-sequence	:	1,	**1**,	**2**,	...	$*$,	$*$,	$*$,

which is impossible.

Hence the only decomposition that will work with corollary 2.11 in [6] occurs when $\alpha = 5$ and

$$h = (1,3,5,\ldots,5,3,1) + (1,0,\ldots,0)$$

is the only possible case, where the second O-sequence must be a Gorenstein sequence of length $s+1$.

Since A is and Artinian algebra with $h - vector = (1,3,5,\ldots)$. There is a homogeneous ideal I in $R = k[x_1, x_2, x_3]$ such that $A = R/I$. By Theorem 3.4, there is an ideal J in R with $I \subset J$ such that $B = R/J$ is a Gorenstein algebra having $(1,3,5,\ldots,5,3,1)$. as its $h - vector$. Then by Theorem 3.4 says that there is an ideal J in R with $I \subset J$

such that $B = k[x_1, x_2, x_3]/\ J$ has h-vector the Gorenstien sequence above.

But by Theorem 2.7 in [11], any Gorenstien ideal having this h-vector is generated in degrees $\leq s-1$. Since I and J agrees in degrees $\leq s-1$ this implies that $J \subset I$, a contradiction. $\square$

Using Proposition 3.5, the following cases in Table 3 are not level.

17)	1, 3, 4, 5, 3, 2	29)	1, 3, 5, 5, 3, 2	51)	1, 3, 6, 5, 3, 2

TABLE 3

Remark 3.6. By the same argument as in Proposition 3.5, one can show that any Artinian algebra with Hilbert function $(1, 3, \ldots, 6, 3, 2)$, cannot be level.

Theorem 3.7 (Theorem 4.1 [1]). *Let $R = k[x_1, x_2, x_3]$ and let $\mathbf{H} = (h_0, h_1, \cdots, h_s)$ be the h-vector of a graded Artinian algebra $A = R/I$ with socle degree s. If*

$$h_{d-1} > h_d \qquad \textit{and} \qquad h_d = h_{d+1} \leq 2d+3,$$

then $\mathbf{H}$ is not level

Using Theorem 3.7, the following 29 cases in Table 4 are not level.

1)	1, 3, 2, 2, 2, 2	2)	1, 3, 3, 2, 2, 2	3)	1, 3, 3, 3, 2, 2
5)	1, 3, 3, 4, 2, 2	9)	1, 3, 4, 2, 2, 2	10)	1, 3, 4, 3, 2, 2
11)	1, 3, 4, 3, 3, 2	12)	1, 3, 4, 4, 2, 2	16)	1, 3, 4, 5, 2, 2
21)	1, 3, 5, 2, 2, 2	22)	1, 3, 5, 3, 2, 2	23)	1, 3, 5, 3, 3, 2
24)	1, 3, 5, 4, 2, 2	26)	1, 3, 5, 4, 4, 2	28)	1, 3, 5, 5, 2, 2
33)	1, 3, 5, 6, 2, 2	38)	1, 3, 5, 7, 2, 2	43)	1, 3, 6, 2, 2, 2
44)	1, 3, 6, 3, 2, 2	45)	1, 3, 6, 3, 3, 2	46)	1, 3, 6, 4, 2, 2
48)	1, 3, 6, 4, 4, 2	50)	1, 3, 6, 5, 2, 2	53)	1, 3, 6, 5, 5, 2
55)	1, 3, 6, 6, 2, 2	60)	1, 3, 6, 7, 2, 2	65)	1, 3, 6, 8, 2, 2
70)	1, 3, 6, 9, 2, 2	75)	1, 3, 6, 10, 2, 2		

TABLE 4

Remark 3.8 (Remark 2.7 [6])**.** If we let $\tilde{h} = (1, \tilde{h}_1, \dots, \tilde{h}_r)$, $r \le s$ be any one of the O-sequences in the decomposition promised by Theorem 3.3, then $\tilde{h}_1 \le h_1$.

Proposition 3.9. *The h-vector* $(1, 3, \dots, 8, 4, 2)$ *is not level.*

Proof. Assume there is an Artinian algebra A with Hilbert function h. By Macaulay's theorem, the only possibilities for α are $3, 4, 5$ and 6. Then, by Theorem 3.4, the vector $(2, 4, 8, \dots, 3, 1)$ is the sum of a Gorenstein sequence $(1, b_1, \dots, b_{s-1}, 1)$ of length $s + 1$ and an O-sequence $(1, b_1, \dots)$, and we have the details as follows.

Case 1.

Gorenstein sequence	:	1,	1,	1,	...	$*$,	1,	1
O-sequence	:	1,	**3**,	**7**,	...	$*$,	$*$,	$*$,

which is impossible.

Case 2.

Gorenstein sequence	:	1,	2,	2,	...	$*$,	2,	1
O-sequence	:	1,	**2**,	**6**,	...	$*$,	$*$,	$*$,

which is impossible.

Case 3.

Gorenstein sequence	:	1,	2,	3,	...	3,	2,	1
O-sequence	:	1,	**2**,	**5**,	...	$*$,	$*$,	$*$,

wihch is impossible.

Case 4.

Gorenstein sequence	:	1,	3,	α,	...	α,	3,	1
O-sequence	:	1,	1,	β,	...	$*$,	$*$,	$*$,

$$\text{where} \quad \alpha \leqq 6, \quad \beta \leqq 1 \quad \Rightarrow \quad \alpha + \beta \leqq 7, \ \text{ but } \ \alpha + \beta = 8,$$

which is a contradiction, as we wished.

□

Remark 3.10. By the same idea as in the proof of proposition 3.9, one can show that the h-vector $(1, 3, \ldots, 9, 4, 2)$ is not level either.

Proposition 3.11. *The h-vector* $(1, 3, 6, \ldots, 4, 3, 2)$ *is not level.*

Proof. Assume there is an Artinian algebra A with Hilbert function h. By Macaulay's theorem, the only possibilities for α are $3, 4, 5$ and

6. Then, by Theorem 3.4, the vector $(2, 3, 4, \ldots, 6, 3, 1)$ is the sum of a Gorenstein sequence $(1, b_1, \ldots, b_{s-1}, 1)$ of length $s+1$ and an O-sequence $(1, b_1, \ldots)$, and we have the details as follows.

Case 1.

Gorenstein sequence	:	1,	1,	1,	...	1,	1,	1
O-sequence	:	1,	2,	**3**,	...	5,	2,	0,

which is impossible.

Case 2.

Gorenstein sequence	:	1,	2,	2,	...	2,	2,	1
O-sequence	:	1,	**1**,	**2**,	...	$*$,	$*$,	$*$,

which is impossible.

Case 3.

Gorenstein sequence	:	1,	2,	3,	...	3,	2,	1
O-sequence	:	1,	1,	1,	...	**3**,	1,	0,

wihch is impossible.

Case 4.

Gorenstein sequence	:	1,	3,	4,	...	4,	3,	1
O-sequence	:	1,	0,	0,	...	2,	0,	0,

which is impossible.

□

Proposition 3.12. *The h-vector* $(1, 3, 5, 4, 3, 2)$ *is not level.*

Proof. Assume there is an Artinian algebra A with Hilbert function h. By Macaulay's theorem, the only possibilities for α are $3, 4, 5$ and

6. Then, by Theorem 3.4, the vector $(2,3,4,5,3,1)$ is the sum of a Gorenstein sequence $(1,b_1,\dots,b_{s-1},1)$ of length $s+1$ and an O-sequence $(1,b_1,\dots)$, and we have the details as follows.

$$(2,3,4,5,3,1)$$

$$\begin{array}{ccc}
(1,2,2,2,2,1) & + & (1,1,2,\dots\ \ ) \\
(1,2,3,3,2,1) & + & (1,1,1,2,\dots) \\
(1,3,3,3,3,1) & + & (1,0,1,\dots\ \ ) \\
(1,3,4,4,3,1) & + & (1,0,0,1,\dots)
\end{array}$$

which are all impossible.

Hence, there is only one way that $(2,3,4,5,3,1)$ can be written as the sum of a Gorenstein sequence of length 6 and an O-sequence , namely $(1,1,1,1,1,1) \ + \ (1,2,3,4,2,0)$.

But $A=(1,3,5,4,3,2), A/I=(1,2,3,4,2,0), I_2 \neq (0), I_3=(0)$ Hence A is not level, as we wished.

$\square$

Proposition 3.13. *The h-vector* $(1,3,5,7,4,2)$ *is not level.*

Proof. Assume there is an Artinian algebra A with Hilbert function h. By Macaulay's theorem, the only possibilities for α are $3,4,5$ and 6. Then, by Theorem 3.4, the vector $(2,4,7,5,3,1)$ is the sum of a Gorenstein sequence $(1, b_1, \dots, b_{s-1}, 1)$ of length $s+1$ and an O-sequence $(1, b_1, \dots)$, and we have the details as follows.

$$\begin{array}{ccl}
(1,1,1,1,1,1) & + & (1,\mathbf{3},\mathbf{6},4,2,0) \\
(1,2,2,2,2,1) & + & (1,2,\mathbf{5},\mathbf{3},1,0) \\
(1,2,3,3,2,1) & + & (1,2,\mathbf{4},\mathbf{2},\dots) \\
(1,3,3,3,3,1) & + & (1,1,\mathbf{4},*,\dots) \\
(1,3,4,4,3,1) & + & (1,1,\mathbf{3},*,\dots) \\
(1,3,5,5,3,1) & + & (1,1,2,0,\dots) \\
(1,3,\mathbf{6},6,3,1) & + & (1,1,1,*,\dots),
\end{array}$$

which are all impossible, as we wished. □

Proposition 3.14. *The h-vector* $(1, 3, 6, 10, 5, 2)$ *is not level.*

Proof. Assume there is an Artinian algebra A with Hilbert function h. By Macaulay's theorem, the only possibilities for α are $3, 4, 5$ and 6. Then, by Theorem 3.4, the vector $(2, 5, 10, 6, 3, 1)$ is the sum of a Gorenstein sequence $(1, b_1, \dots, b_{s-1}, 1)$ of length $s+1$ and an O-sequence $(1, b_1, \dots)$, and we have the details as follows.

Case 1.

Gorenstein sequence	:	1,	1,	1,	1,	1,	1
O-sequence	:	1,	4,	$*$,	$*$,	$*$,	$*$,

which is impossible.

Case 2.

Gorenstein sequence	:	1,	2,	2,	2,	2,	1
O-sequence	:	1,	3,	8,	$*$,	$*$,	$*$,

which is impossible.

Case 3.

Gorenstein sequence	:	1,	2,	3,	3,	2,	1
O-sequence	:	1,	3,	7,	,		

wihch is impossible.

Case 4.

Gorenstein sequence	:	1,	3,	3,	3,	3,	1
O-sequence	:	1,	2,	7,	$\ldots,$		

wihch is impossible.

Case 5.

Gorenstein sequence	:	1,	3,	4,	4,	3,	1
O-sequence	:	1,	2,	6,	$\ldots,$		

wihch is impossible.

Case 6.

Gorenstein sequence	:	1,	3,	5,	5,	3,	1
O-sequence	:	1,	2,	5,	$\ldots,$		

wihch is impossible.

$\square$

Definition 3.15. (a) A total order on the monomials of each degree is said to be a ***term order*** if

i) $x_1 > \cdots > x_n$, and

ii) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m, m_1 and m_2.

(b) The ***reverse lexicographic order*** is a term order defined to be $x_1^{i_1} \dots x_n^{i_n} > x_1^{j_1} \dots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $t > s$ and $i_s < j_s$.

(c) The ***lexicographic order*** is a term order defined to be $x_1^{i_1} \dots x_n^{i_n} > x_1^{j_1} \dots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $t < s$ and $i_s > j_s$.

(d) Let S be a set of monomials in R_d. S is ***lex-segment*** if a monomial m of deree d is in S, then every monomial m' of degree d in R_d with $m' > m$ with respect to lexicographic order belongs to S.

(e) Let $I = \bigoplus_{t \geq 0} I_t$ be a homogeneous ideal of R. We say that I is a ***lex-segment ideal*** if for every $t \geq 0$, I_t is (as a vector space) lex-segment.

Remark 3.16. (Non-Cancellable Case) Let I be a lex-segment ideal in R with Hilbert function $h = (1, 3, 6, 10, 4, 2)$. Then the minimal free resolution of R/I is

$$\begin{array}{llllll} 0 & \rightarrow & \boldsymbol{R^6}(\boldsymbol{-6}) \oplus R^2(-7) \oplus R^2(-8) & \rightarrow & R^{16}(-5) \oplus \boldsymbol{R^4}(\boldsymbol{-6}) \oplus R^4(-7) \\ & \rightarrow & R^{11}(-4) \oplus R^2(-5) \oplus R^2(-6) & \rightarrow & R \quad \rightarrow \quad R/I \quad \rightarrow \quad 0\,, \end{array}$$

and hence the copies $R(-6)$ in the last free module are not cancelable, that is, any Artinian algebra with Hilbert Function h has at least 2 dimensional socle elements in degree 3. Hence the case 77) $(1, 3, 6, 10, 4, 2)$ cannot be level.

4. **Macaulay's Inverse System**

We now recall an interesting method for construsting Artinian level algebras. This method is based on the idea of ***Macaulay's Inverse Systems***. We will only give a quick review of the method, and refer the reader [5] in the book for more details.

Let $R = k[x_1, \cdots, x_n]$ and $S = k[y_1, \cdots, y_n]$. We can consider S as a graded R-module by: if $F \in S_j$ then $x_i \circ F = \frac{\partial}{\partial y_i} F$. We extend this action in the obvious way, and note that the action *lowers* the degree of a given form on S and hence S is not a finitely generated R-module.

There is an inclusion reversing function from the ideals of R to the R-submodules of S defined by:

$$\varphi_1 : \{\text{ideals of } R\} \to \{R\text{-submodules of } S\}$$

where

$$\varphi_1(I) = \{F \in S | G \circ F = 0 \text{ for all } G \in I\}$$

This is a $1-1$ correspondence whose inverse φ_2 is given by $\varphi_2(M) = \mathrm{ann}_R(M) = \{r \in R \mid r \cdot x = 0, \forall x \in M\}$. In fact, we denote $\varphi_1(I)$ by I^{-1}, which is called the ***inverse system*** to I.

It is very easy to construct I^{-1} (and this is at the heart of the proof of the 1-1 correspondence). First observe that the pairing

$$R_j \times S_j \longrightarrow S_0 \simeq k$$

is a perfect pairing, and so S_j can be identified with R_j^* (the dual vector space to R_j). If V is a subspace of R_j we write $V^\perp$ for the annihilator of V in this pairing. If $I \subset R$ is an ideal and I_j its j^{th} graded piece, then Macaulay observed that:

$$(I^{-1})_j = I_J^\perp.$$

It follows immediately that

$$\dim_k (I^{-1})_j = \dim_k R_j - \dim_k I_j = \mathbf{H}(R/I, j).$$

It is a simple consequence of this last observation that I^{-1} is a finitely generated R-submodule of S if and only if R/I is Artinian.

Remark 4.1. There is another way to define Inverse Systems which considers S as an R-module in a different way. In this other method, we consider the ***contraction*** operations, D_{x_i} where, if F is a monomial in S_j then

$$D_{x_i}(F) = \begin{cases} 0, & \text{if } y_i \text{ does not divide } F, \\ F/y_i & \text{if } y_i \text{ divides } F. \end{cases}$$

We extend this action to all of S in the obvious way and recall that when the characteristic of k is 0, this action is equivalent to the one described above. The contraction operation has the advantage that it doesn't end up increasing the sizes of coefficients (see [5] for more details).

The really interesting connection between inverse systems and what we've been considering is the following theorem of Macaulay. We continue with notations as above.

Theorem 4.2. *Let I be an Artinian ideal of R and I^{-1} its inverse system. Then I^{-1} has exactly ν_j minimal generators of degree j if and only if the socle of R/I in degree j has dimension exactly ν_j.*

Remark 4.3. 1) This gives us a new interpretation of the socle vector of an Artinian algebra of the form $A = k[x_1, \cdots, x_n]/I$. The entries of the socle vector tell us the number of generators of the inverse system of I in each degree.

2) Since we are interested in level algebras (Atrinian, say, with socle degree s, type c and embedding dimension n) then this theorem tells us how to make ***all*** of them. We look at every subspace of $S_s = k[y_1, \cdots, y_n]_s$ of dimension c and form the R-submodule of S generated by that subspace. The result is a level algebra of the type we are looking for and every level algebra of socle degree s, type c and embedding dimension n arises in this way.

Example 4.4. Suppose we would like to construct a level algebra with socle degree 4, embedding dimension 3 and type 2. Macaulay's Theorem says we have to look at a two dimensional vector space of S_4, where $S = k[y_1, y_2, y_3]$ and take the inverse system it generates.

For example, consider the vector space of S_4 generated by $F_1 = y_2^5 + y_3^5$ and $F_2 = y_1 y_2 y_3^3$. The inverse system, call it M generated by these two elements of degree 5 will have

$$\begin{aligned}
M_4 &= \langle y_2^4, y_3^4, y_2 y_3^3, y_1 y_3^3, y_1 y_2 y_3^2 \rangle, \\
M_3 &= \langle y_2^3, y_3^3, y_1 y_3^2, y_2 y_3^2, y_1 y_2 y_3 \rangle, \\
M_2 &= \langle y_2^2, y_3^2, y_1 y_2, y_2 y_3, y_3 \rangle, \\
M_1 &= \langle y_1, y_2, y_3 \rangle, \quad \text{and} \\
M_0 &= \langle 1 \rangle.
\end{aligned}$$

So, if $I = \operatorname{ann}_R(M)$ and $A = k[x_1, x_2, x_3]/I$ then we obtain a level h-vector of A

$$(1, 3, 5, 5, 5, 2).$$

Now we will prove that the case 20) cannot be level and we need the following lemma to prove it first.

Lemma 4.5. *Let $R = k[x, y, z]$. If I is a homogeneous ideal of R such that R/I has the Hilbert function $\mathbf{H} = (1, 3, 4, 5, \dots)$, then two minimal generators of I in degree 2 have a linear common factor.*

Proof. Let I be an ideal in $R = k[x, y, z]$ so that the Hilbert function of $A = R/I$ begins

$$1 \quad 3 \quad 4 \quad 5 \quad \dots .$$

Let $\langle F, G\rangle = I_2$ and assume that F, G is a regular sequence. Since 4 and 5 have a maximal growth in degrees 2 and 3, we see that I does not have any generators in degree 3, that is,

$$\begin{aligned} R_1 I_2 &= I_3 \\ &= \langle Fx, Fy, Fz, Gx, Gy, Gz\rangle. \end{aligned}$$

Since $\dim_k I_3 = 5 = 10 - \mathbf{H}(R/I, 3)$, we have that one of Fx, Fy, Fz, Gx, Gy and Gz is a linear combination of the rest of 5 elements.

Without loss of generality, we may assume that

$$Fx \in \langle Fy, Fz, Gx, Gy, Gz\rangle.$$

Then

$$Fx = aFy + bFz + cGx + dGy + eGz$$

where a, b, c, d, $e \in k$, and so

$$F \cdot (x - ay - bz) = G \cdot (cx + dy + ez) \in (G)$$

Since F, G is a regular sequence, we have that

$$(x - ay - bz) \in (G),$$

which is a contradiction since $\deg(G) = 2$. In other words, F and G cannot be a regular sequence.

Note that $k[x, y, z]$ is a unique factorization domain. Hence

$$(x - ay - bz) \mid F \cdot (x - ay - bz)$$

implies

$$(x - ay - bz) \mid G \quad \text{or} \quad (x - ay - bz) \mid (cx + dy + ez).$$

However, if $(x - ay - bz) \mid (cx + dy + ez)$, then $F = \alpha \cdot G$ for some $\alpha \in k - (0)$, that is, $\dim_k \langle F, G \rangle = 1$, a contradiction.

Thus $(x - ay - bz) \mid G$, that is, $G = (x - ay - bz) \cdot L$ for some linear form $L \in R_1$ and so

$$F \cdot (x - ay - bz) = G \cdot (cx + dy + ez)$$

$$= (cx + dy + ez) \cdot L \cdot (x - ay - bz)$$

This implies that $F = (cx + dy + ez) \cdot L$.

In other words, F and G have a linear common factor $L \in R_1$, as we wished. $\square$

Proposition 4.6. *Any Artinian algebra with Hilbert function* $(1, 3, 4, 5, \ldots, 6, 2)$ *is not level.*

Proof. First of all, we will prove that $(1,3,4,5,6,2)$ cannot be level. If it were, let I be a level Artinian ideal in $R{=}k[x,y,z]$ having the Hilbert function $h{=}(1,3,4,5,6,2)$. Then we observe, from the Hilbert function, that the forms in I_2 have to have a common factor and that I has no generators in degrees 3 and 4. I.e., we have that $(I_{\leq 4}){=}(x^2,xy)$ or (xy,xz).

Case 1. $(I_{\leq 4}){=}(x^2,xy)$

Let $S{=}k[X,Y,Z]$ and $F,G\in\langle F,G\rangle^{\perp}=I$. Since $(I_{\leq 4}){=}(x^2,xy)$, we may assume that, $F,G\in\langle Y^5,Y^4Z,Y^3Z^2,Y^2Z^3,YZ^4,Z^5,XZ^4\rangle$.

So let

$$F=a_1Y^5+a_2Y^4Z+a_3Y^3Z^2+a_4Y^2Z^3+a_5YZ^4+a_6Z^5+aXZ^4,$$
$$G=b_1Y^5+b_2Y^4Z+b_3Y^3Z^2+b_4Y^2Z^3+b_5YZ^4+b_6Z^5.$$

Since $H(A,1)=3$, we cannot have all F and G in $k[Y,Z]_6$ and so we may assume that $a=1$ and G in $k[Y,Z]_6$.

Now consider all the contradictions of F and G. They may be viewed as vectors in k^7 as follows:

$$\begin{array}{rcl} D_x(F) & = & Z^4 \\ & \leftrightarrow & (0,0,0,0,1,0), \\ D_y(F) & = & a_1Y^4 + a_2Y^3Z + a_3Y^2Z^2 + a_4YZ^3 + a_5Z^4 \\ & \leftrightarrow & (a_1,a_2,a_3,a_4,a_5,0), \\ D_z(F) & = & a_2Y^4 + a_3Y^3Z + a_4Y^2Z^2 + a_5YZ^3 + a_6Z^4 + XZ^3 \\ & \leftrightarrow & (a_2,a_3,a_4,a_5,a_6,1), \\ D_x(G) & = & 0 \\ & \leftrightarrow & (0,0,0,0,0,0), \\ D_y(G) & = & b_1Y^4 + b_2Y^3Z + b_3Y^2Z^2 + b_4YZ^3 + b_5Z^4 \\ & \leftrightarrow & (b_1,b_2,b_3,b_4,b_5,0), \\ D_z(G) & = & b_2Y^4 + b_3Y^3Z + b_4Y^2Z^2 + b_5YZ^3 + b_6Z^4 \\ & \leftrightarrow & (b_2,b_3,b_4,b_5,b_6,0). \end{array}$$

Hence

$$\text{A}= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & 1 \\ b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & 0 \end{pmatrix}$$

has a rank at most 5. However, since $H(A,4) = 6$, we have to have rank(A)=6, which is impossible.

Case 2. $(I_{\leq 4})$=(xy, xz)

$$F = a_1Y^5 + a_2Y^4Z + a_3Y^3Z^2 + a_4Y^2Z^3 + a_5YZ^4 + a_6Z^5 + X^5,$$

$$G = b_1Y^5 + b_2Y^4Z + b_3Y^3Z^2 + b_4Y^2Z^3 + b_5YZ^4 + b_6Z^5.$$

Then,

$$\begin{array}{rcl}
D_x(F) & = & X^4 \\
& \leftrightarrow & (0,0,0,0,1,0), \\
D_y(F) & = & a_1Y^4 + a_2Y^3Z + a_3Y^2Z^2 + a_4YZ^3 + a_5Z^4 \\
& \leftrightarrow & (a_1,a_2,a_3,a_4,a_5,0), \\
D_z(F) & = & a_2Y^4 + a_3Y^3Z + a_4Y^2Z^2 + a_5YZ^3 + a_6Z^4 \\
& \leftrightarrow & (a_2,a_3,a_4,a_5,a_6,0), \\
D_x(G) & = & 0 \\
& \leftrightarrow & (0,0,0,0,0,0), \\
D_y(G) & = & b_1Y^4 + b_2Y^3Z + b_3Y^2Z^2 + b_4YZ^3 + b_5Z^4 \\
& \leftrightarrow & (b_1,b_2,b_3,b_4,b_5,0), \\
D_z(G) & = & b_2Y^4 + b_3Y^3Z + b_4Y^2Z^2 + b_5YZ^3 + b_6Z^4 \\
& \leftrightarrow & (b_2,b_3,b_4,b_5,b_6,0).
\end{array}$$

Hence

$$\text{A}= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ b_2 & b_3 & b_4 & b_5 & b_6 & 0 \end{pmatrix}$$

has a rank at most 5. By the same reason as Case 1, $H(A,4) \leq 5$, which is a contradiction, since $H(A,4) = 6$.

By Cases 1 and 2, the h-vector (1, 3, 4, 5, 6, 2) cannot be level.

$\square$

Moreover, using the same idea, we see that any h-vector

$$(1,3,5,\ldots,6,2)$$

cannot be level in general, as we wished.

5. **Some Construction of Gorenstein and Level Artinian O-sequences**

Definition 5.1 (Definition 3.1 [12])**.** (a) A finite set $\mathbb{X}$ of points $\mathbb{P}^2$ is called a ***basic configuration*** of type (d, e) if there exist distinct elements b_j, c_j in k such that

$$I_{\mathbb{X}} = \left(\prod_{j=1}^{d} (x - b_j z), \prod_{j=1}^{e} (y - c_j z) \right).$$

We denote $\mathbb{X} := \mathbb{B}(d, e)$.

(b) A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a ***pure configuration*** if there exist finite basic configurations $\mathbb{B}(d_1, e_1), \ldots, \mathbb{B}(d_m, e_m)$ where $e_1 > \cdots > e_m$, which satisfy the following three conditions:

(i) $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \varnothing$ if $i \neq j$,

(ii) $\mathbb{X} = \mathbb{B}(d_1, e_1) \bigcup \cdots \bigcup \mathbb{B}(d_m, e_m)$,

(iii) $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$ for all $1 \leq i \leq m-1$, where $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \to \mathbb{P}^1$ is the map defined by sending the point (x, y, z) to the point (y, z).

In this case, we denote $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$.

Theorem 5.2 (Corollary 3.10, [10])**.** *Let $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ be a pure configuration in $\mathbb{P}^2$. Then $\mathbb{X}$ is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all $1 \leqslant i \leqslant m-1$.

If $r(A)$ is Cohen-Macaulay type of a Cohen-Macaulay standard graded k-algebra A, and if $A = \bigoplus_{i\geq 0} A_i$ is an Artinian level algebra, then $r(A) = \dim_k A_{\sigma(A)-1}$, where $\sigma(A) = \min\{i | A_i = 0\}$. If $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ is a level pure configuration in $\mathbb{P}^2$, then $r(\mathbb{Z}) = r(R/I_{\mathbb{Z}}) = m$.

Lemma 5.3 (Lemma 3.14, [10])**.** *Let $\mathbb{Z}$ be a level set of points in $\mathbb{P}^n$ and $\mathbb{X}$ a subset of $\mathbb{Z}$. Set $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is an Artinion level graded k-algebra with $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \sigma(\mathbb{Z}) - 1$ and $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq r(\mathbb{Z})$.*

Corollary 5.4 (Corollary 3.15, [10])**.** *Let $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ be a level pure configuration in $\mathbb{P}^2$ and $\mathbb{X}$ a subset of $\mathbb{Z}$. Set $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. Then $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is an Artinion level graded k-algebra with $\sigma(R/(I_{\mathbb{X}}+I_{\mathbb{Y}})) = d_1 + e_1 - 2$ and $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq m$.*

In the following example, we introduce how to produce all level O-sequences among 79 cases in Table 1 using Link-Sum method. Moreover, we use the Computer programs CoCoA [3] and Splus for these constructions.

Example 5.5. Now we can construct 23 cases using Corollary 5.4 among 79 cases in Table 1, which are: 4), 13), 14), 18), 19), 30), 31), 35), 36), 37), 41), 42), 57), 58), 59), 62), 63), 64), 68), 69), 73), 74), 79).

4)

$$\mathbb{Z} = \begin{cases} * & \bullet & \\ \bullet & * & \bullet \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 3 & 3 & 3 & 3 & 3 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 12 & 14 & 14 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 3 & 3 & 3 & 2 & 0 & \rightarrow . \end{array}$$

13)

$$\mathbb{Z} = \begin{cases} * & * & \\ * & * & * \\ * & * & * \\ * & * & \bullet \\ * & * & \bullet \\ * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 4 & 4 & 4 & 4 & 4 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 11 & 13 & 13 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 4 & 4 & 3 & 2 & 0 & \rightarrow . \end{array}$$

14)

$$\mathbb{Z} = \begin{cases} * & \bullet & \\ \bullet & * & \bullet \\ * & \bullet & * \\ * & * & * \\ * & * & * \\ * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 4 & 4 & 4 & 4 & 4 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 12 & 13 & 13 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 4 & 4 & 4 & 2 & 0 & \rightarrow . \end{array}$$

18)

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & \\ \bullet & * & * \\ \bullet & * & * \\ \bullet & * & * \\ * & * & * \\ * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 4 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 11 & 12 & 12 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 4 & 5 & 4 & 2 & 0 & \rightarrow . \end{array}$$

19)

$$\mathbb{Z} = \begin{cases} * & * & & \\ * & * & & \\ * & * & * & * \\ * & * & * & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 20 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 4 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 14 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 4 & 5 & 5 & 2 & 0 & \rightarrow . \end{array}$$

30)

$$\mathbb{Z} = \begin{cases} \bullet & * & \\ * & * & * \\ * & * & * \\ * & * & \bullet \\ * & * & \bullet \\ \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 5 & 5 & 5 & 5 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 11 & 12 & 12 & \rightarrow & \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 5 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

31)

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & \\ \bullet & \bullet & * \\ \bullet & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 5 & 5 & 5 & 5 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 12 & 12 & 12 & \rightarrow & \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 5 & 5 & 2 & 0 & \rightarrow & . \end{array}$$

35)

$$\mathbb{Z} = \begin{cases} * & * & \\ * & * & * \\ \bullet & * & * \\ \bullet & * & * \\ \bullet & * & \bullet \\ \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 6 & 6 & 6 & 6 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 10 & 11 & 11 & \rightarrow & \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 6 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

36)

$$\mathbb{Z} = \begin{cases} * & * & \\ * & * & * \\ \bullet & * & * \\ \bullet & * & * \\ \bullet & \bullet & * \\ \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 6 & 5 & 2 & 0 & \rightarrow . \end{array}$$

37)

$$\mathbb{Z} = \begin{cases} * & \bullet & * & & \\ * & * & * & & \\ \bullet & * & * & * & * \\ \bullet & \bullet & * & * & * \\ \bullet & \bullet & * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 19 & 21 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 6 & 6 & 2 & 0 & \rightarrow . \end{array}$$

41)

$$\mathbb{Z} = \begin{cases} * & * & \\ * & * & * \\ \bullet & * & * \\ \bullet & \bullet & * \\ \bullet & \bullet & * \\ \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 7 & 5 & 2 & 0 & \rightarrow . \end{array}$$

42)

$$\mathbb{Z} = \begin{cases} * & * & & \\ \bullet & * & & \\ * & \bullet & * & * \\ * & \bullet & \bullet & * \\ * & * & \bullet & \bullet \\ * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 20 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 5 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 13 & 13 & 13 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 5 & 7 & 6 & 2 & 0 & \rightarrow . \end{array}$$

57)

$$\mathbb{Z} = \begin{cases} * & & \\ * & & \\ * & * & * \\ \bullet & * & \bullet \\ * & \bullet & \bullet \\ * & * & \bullet \\ * & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 10 & 11 & 11 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 6 & 4 & 2 & 0 & \rightarrow . \end{array}$$

58)

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & \\ \bullet & \bullet & \bullet \\ \bullet & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 6 & 5 & 2 & 0 & \rightarrow & . \end{array}$$

59)

$$\mathbb{Z} = \begin{cases} * & \bullet & \bullet \\ * & * & \bullet \\ * & * & * & \bullet & \bullet \\ * & * & * & * & * \\ * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 15 & 19 & 21 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 6 & 6 & 2 & 0 & \rightarrow & . \end{array}$$

62)

$$\mathbb{Z} = \begin{cases} * \\ * \\ * \\ \bullet & * & * & * \\ * & * & \bullet & \bullet \\ * & \bullet & \bullet & * \\ * & \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc} \mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 17 & 19 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 11 & 12 & 12 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 7 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

63)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * & * \\ * & * & * \\ * & * & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rccccccccc}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 7 & 5 & 2 & 0 & \rightarrow .
\end{array}$$

64)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & * \\ * & \bullet & * & \bullet \\ * & \bullet & * & \bullet \\ \bullet & \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rccccccccc}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 20 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 13 & 13 & 13 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 7 & 6 & 2 & 0 & \rightarrow .
\end{array}$$

68)

$$\mathbb{Z} = \begin{cases} \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & * \\ \bullet & * & * \\ * & * & * \\ * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rccccccccc}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 9 & 12 & 15 & 17 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 8 & 8 & 8 & 8 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 8 & 5 & 2 & 0 & \rightarrow .
\end{array}$$

69)

$$\mathbb{Z} = \begin{cases} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet & \bullet & * \\ \bullet & * & * & * \\ * & * & * & * \\ * & * & * & * \end{cases}$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 20 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 8 & 8 & 8 & 8 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 12 & 12 & 12 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 8 & 6 & 2 & 0 & \rightarrow .
\end{array}$$

73)

$$\mathbb{Z} = \left\{ \begin{array}{cccc}
\bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & * \\
\bullet & \bullet & * & * \\
\bullet & * & * & * \\
* & * & * & *
\end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 17 & 19 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 9 & 5 & 2 & 0 & \rightarrow .
\end{array}$$

74)

$$\mathbb{Z} = \left\{ \begin{array}{cccc}
\bullet & \bullet & & \\
\bullet & \bullet & & \\
* & * & \bullet & \bullet \\
* & * & \bullet & \bullet \\
* & * & * & \bullet \\
* & * & * & *
\end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcrrrrrrrl}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 20 & \rightarrow \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 11 & 11 & 11 & \rightarrow \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 9 & 6 & 2 & 0 & \rightarrow .
\end{array}$$

79)

$$\mathbb{Z} = \left\{ \begin{array}{cccc}
\bullet & \bullet & & \\
\bullet & \bullet & & \\
* & \bullet & \bullet & \bullet \\
* & * & * & \bullet \\
* & * & \bullet & \bullet \\
* & * & * & *
\end{array} \right.$$

Then the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, and $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are

$$\begin{array}{rcccccccccc}
\mathbf{H}_{\mathbb{Z}} & : & 1 & 3 & 6 & 10 & 14 & 18 & 20 & \rightarrow & \\
\mathbf{H}_{\mathbb{X}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow & \\
\mathbf{H}_{\mathbb{Y}} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow & \\
\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} & : & 1 & 3 & 6 & 10 & 6 & 2 & 0 & \rightarrow & .
\end{array}$$

Remark 5.6. We have classified 78 cases among 79 cases in Table1. In other words, we have only one case 52) $(1, 3, 6, 5, 4, 2)$ left to be proved.

However, we will omit this non-level case, which is shown in Example 5.7 and Remark 2.12 in [6].

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ABSTRACT

Some Construction of All Level Artinian O-sequences of Socle Degree 5 and Type 2

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The purpose of this study was to examine the Codimension 3, Socle Degree 5 and Type 2 Artinian O-sequences 79 based on Theorem of Fröberg and Laksov.

First of all, we prove that 53 sequences cannot be level using [1], [6], [11].

Then we introduce how to construct 23 sequences using CoCoA, S-plus.