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Some Algorithms for Artinian O-sequences

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성신여자대학교 교육대학원

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이 논문을 석사학위논문으로 제출함.

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윤영주의 석사학위 논문으로 인준함.

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논 문 개 요

본 논문은 여차원이 3인 O-sequence 중 그 값이 증가하다 일정하게 유지되고 다시 증가하는 경우, 즉 h_0 h_1 ... h_{d-1} h_d h_{d+1} 이고 $h_{d-1} = h_d < h_{d+1}$ 을 만족하는 경우의 level 여부를 연구하고 있다.

본 연구에 필요한 수열을 출력해주는 두 개의 코코아 알고리즘을 작성하여 예제와 함께 소개하였으며, 이 알고리즘의 결과로 얻어진 수열의 minimal free resolution을 토대로 level이 되지 않는 수열을 증명하였다.

CONTENTS

1. **Introduction**

If $R = k[x_0, x_1, \dots, x_n] = \oplus_{i\geq 0} R_i$, where k is an algebraically closed field of characteristic 0, and if I is a homogeneous ideal of R, and $A = R/I$, then the *Hilbert function of* A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

In case I is the ideal of a subscheme, $\mathbb{X}$ of $\mathbb{P}^n$, the Hilbert function of $A = R/I$ is sometimes denoted by $\mathbf{H}_{\mathbb{X}}(-)$.

It is worth noting that R is a *standard* graded algebra since $R = k[R_1]$, that is, R is generated (as an k-algebra) by its piece of degree 1. If I is a homogeneous ideal of R, then R is again a standard graded k-algebra. Furthermore, if I has height $n+1$ in R, then $A = R/I$ is an *Artinian* k-algebra, and hence $\dim_k A < \infty$. Thus we can write $A = k \oplus A_1 \oplus \cdots \oplus A_s$ where $A_s \neq 0$. We call s the socle degree of A.

We associate to the graded Artinian algebra A a vector of non-negative integers which is an $(s+1)$-tuple, called the h-vector of A and denoted $h(A)$. Let A be as above. Then it is defined as follows.

$$h(A) := (1, \dim_k A_1, \dots, \dim_k A_s) = (h_0, h_1, \dots, h_s) \quad \text{with} \quad h_s \neq 0.$$

Let h and i be positive integers. Then h can be written uniquely in the form

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expansion for h is called the *i-binomial expansion* of h. Also, define

$$h^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_{i-1}+1}{(i-1)+1} + \cdots + \binom{m_j+1}{j+1},$$

and $0^{\langle i \rangle} = 0$.

In Section 2, we give some conjecture if a given O-sequence of codimension 3 is not level based on the result of [4]. In Proposition 2.6, we find a partial answer to the conjecture.

In [4], they showed that some O-sequences are not level if they have a flat part in the middle. More precisely, an O-sequence $h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d$ with $h_{d-1} > h_d$ is not level with some condition (see Proposition 2.4). However, we have not seen any level O-sequence $h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_{d+1}$ with $h_{d-1} = h_d < h_{d+1}$ of codimension 3. In Section 3, we obtain some result about this case. Moreover, we give a description of a k-configuration (see [2], [5], and [6] for the definition) in $\mathbb{P}^2$ and $\mathbb{P}^3$ whose minimal free resolution has a non-cancelable Betti number in the last free module. In fact, those properties are true in $\mathbb{P}^n$ in general when $n \geq 4$.

2. **Some Conjectures on Artinian Algebras of Codimension 3**

Definition 2.1 (Definition 2.1 [4])**.** *i)* The sequence $\{h_i\}_{i\geq 0}$ (with $h_0 = 1$ and $h_1 \leq n$) is called an *O-sequence* if there is a homogeneous ideal $I \subset R$ such that if $A = R/I$ then $\mathbf{H}_A(i) = h_i$.

ii) In particular, the vector $h = (1, n, h_2, \dots, h_s)$ is called an *O-sequence* if there is an Artinian quotient A of R whose *h-vector* is h.

iii) The vector $h = (1, n, h_2, \dots, h_s)$ is called a *level sequence* if there is a level Artinian algebra having h as its h-vector. Moreover, we say that the sequence is a *Gorenstein sequence* if it is a level sequence with $h_s = 1$.

When $h_s \neq 0$ we say that $s+1$ is the *length* of the sequence.

Definition-Proposition 2.2 (Definition-Proposition 2.21, [3])**.** Let $R = k[x_0, \dots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. Then

(a) If $B = B_0 \oplus \cdots \oplus B_\ell$ ($B_\ell \neq 0$) is an Artinian algebra, then B is *level* if and only if $B_\ell = \operatorname{Ann}(B_1)$.

(b) A is a *level algebra* if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. $\operatorname{rank} \mathcal{F}_{n-(d-1)} =$ Cohen-Macaulay type of A.

(c) If $\overline{L}$ is a linear non-zero divisor in $A = R/I$, then A is level if and only if $A/\overline{L}A \simeq A/(L, I_{\mathbb{X}})$ is level.

(d) i) If $\mathbb{X}$ is a non-degenerate set of points in $\mathbb{P}^n$, $A = R/I_{\mathbb{X}}$ its coordinate ring, then we say that ℓ is the *socle degree* of $\mathbb{X}$ if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where $\overline{L}$ is any linear non-zero-divisor of A.

ii) $\mathbb{X}$ is called a *level set* of points if $A = R/I_{\mathbb{X}}$ is a level algebra. In this case, the socle degree of $\mathbb{X}$ is $\ell = \sigma(\mathbb{X}) + n - 1$.

(e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an Artinian algebra) $b = \{b_i\}_{i \geq 0}$ with $b_1 = n + 1$, is called *level* if there is a level set of points in $\mathbb{P}^n$ with Hilbert function b.

Theorem 2.3 (Theorem 2.17, [4])**.** *Let $h_{d-2},\ h_{d-1},\ h_d$ be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d-1 \rangle} \qquad \text{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2 \rangle}\ .$$

Let I be any ideal in $R = k[x_1, \ldots, x_n]$ such that the Hilbert function of R/I satisfies

$$\begin{array}{lcl} \mathbf{H}(R/I, d-2) & = & h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) & = & h_{d-1}, \\ \mathbf{H}(R/I, d) & = & h_d \, . \end{array}$$

Then, the ring R/I has socle of dimension ε in degree $d-2$. Consequently, if I has minimal free resolution $\mathbb{F}$ as above, then

$$\beta_{n-1,d+n-2} = \varepsilon.$$

Proposition 2.4 (Proposition 2.21, [4])**.** *Let $h = (1, n, h_2, \ldots, h_s)$ be the h-vector of an Artinian algebra with socle degree s. Then h is **not** a level sequence for each of the following cases:*

a) $h_d = h_{d+1} = p \leq d - 1$ *and* $h_{d-1} > p$*;*

b) $h_d = h_{d+1} = p = d$ *and* $h_{d-1} > p = d$*;*

c) $h_d = h_{d+1} = p = d + 1$ *and* $h_{d-1} > d + 1 = p$*;*

d) $h_d = h_{d+1} = p \leq 2d$ *and* $h_{d-1} \geq p + n$ *and* $d \geq n + 2$*.*

Example 2.5. Consider an Artinian O-sequence $\mathbf{H}$: 1 3 6 10 6 6. Let I be the lex-segment ideal in $R = k[x, y, z]$ with Hilbert function $\mathbf{H}$. Then the minimal free resolution of R/I is

$$\begin{array}{lllll} 0 & \to & R^5(-6) \oplus R(-7) \oplus R^6(-8) & \to & R^{13}(-5) \oplus R^2(-6) \oplus R^{13}(-7) \\ & \to & R^9(-4) \oplus R(-5) \oplus R^7(-6) & \to & R \;\to\; R/I \;\to\; 0. \end{array}$$

Hence all five copies $R(-6)$ in the last free module cannot be canceled, and thus $\mathbf{H}$ is not level. However, this case cannot be covered by Proposition 2.4.

Proposition 2.6. *Let $\mathbf{H} = \{h_i\}_{i\geq 0}$ be an O-sequence with*

$$h_{d-1} > h_d, \quad h_d \leq d+1, \quad h_{d+1} \geq d+1, \quad \ldots$$

for some $i \geq 1$. Then any graded ring with Hilbert function $\mathbf{H}$ is not level.

Proof. Note that, by the proof of Proposition 2.4, any graded ring with Hilbert function

$$\mathbf{H}' \quad : \quad 1 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d \quad \rightarrow$$

has a socle element in degree $d-1$.

Now let $A = \oplus_{i\geq 0} A_i$ be a graded ring with Hilbert function

$$\mathbf{H} \quad : \quad 1 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_{d+1} \quad \cdots$$

Let $A_{d+1} = \langle f_1, f_2, \ldots, f_{h_{d+1}} \rangle$ and $I = \langle f_{h_d+1}, \cdots, f_{h_{d+1}} \rangle \oplus_{j\geq d+2} A_j$. Then a graded ring $B = A/I$ has the Hilbert function $1 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d$, and hence B has a socle element in degree $d-1$ by Proposition 2.4, and so does A since $A_i = B_i$ for every $i \leq d$, as we wished. □

3. **Some Algorithms for Obtaining Codimension3 Artinian O-sequences**

In this section, we are interested in the following type of Hilbert function of codimension 3:

$$h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_{d+1} \quad \cdots$$

with $h_{d-1} = h_d < h_{d+1}$.

Example 3.1. Consider an O-sequence $\mathbf{H}$: 1 3 4 4 5. Let I be a lex-segment ideal in $R = k[x, y, z]$. Then the minimal free resolution of R/I is

$$\begin{array}{lllll} 0 & \rightarrow & R(-5) \oplus R^5(-7) & \rightarrow & R(-3) \oplus R^2(-4) \oplus R^{11}(-6) \\ & \rightarrow & R^2(-2) \oplus R(-3) \oplus R^6(-5) & \rightarrow & R \ \rightarrow \ R/I \ \rightarrow \ 0. \end{array}$$

By the same idea as in Example 2.5, $\mathbf{H}$ is not level. This example gives us the following question.

Question 3.2. Let $\mathbf{H} = (h_0, h_1, \ldots, h_{d-2}, h_{d-1}, h_d)$ such that $h_0 \le h_1 \le \cdots \le h_{d-2} = h_{d-1} < h_d$.

(a) Let A be a graded ring with Hilbert function $\mathbf{H}$. Does the minimal free resolution of A have a non-cancelable Betti number in the last free module?

(b) Is $\mathbf{H}$ *not* a level O-sequence?

We will give a partial answer to Question 3.2 with some condition, that is, a generic Hilbert function in Remark 3.3.

Remark 3.3. By Theorem 2.3 and the same argument as in Example 2.5, we see that the following two Artinian O-sequences $\mathbf{H}_1 : \ 1 \ \ 3 \ \ 3 \ \ 4$ and $\mathbf{H}_2 : \ 1 \ \ 3 \ \ 6 \ \ 6 \ \ 7$ are not level. In fact, if I and J are lex-segment ideals of R with Hilbert functions $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively, then the minimal free resolutions of R/I and R/J are

$$\begin{array}{ccccccccc} 0 & \to & R(-4) \oplus R^4(-6) & \to & R^3(-3) \oplus R^9(-5) & & & & \\ & \to & R^3(-2) \oplus R^5(-4) & \to & R & \to & R/I & \to & 0. \end{array}$$

$$\begin{array}{ccccccccc} 0 & \to & R(-5) \oplus R^7(-7) & \to & R^4(-4) \oplus R^{15}(-6) & & & & \\ & \to & R^4(-3) \oplus R^8(-5) & \to & R & \to & R/I & \to & 0. \end{array}$$

By the same idea as in Example 2.5, $\mathbf{H}_1$ and $\mathbf{H}_2$ are not level.

What can we say about the O-sequence $\mathbf{H} \ : \ 1 \ \ 3 \ \ 6 \ \ \cdots \ \ 6 \ \ 7$?

If the socle degree of $\mathbf{H}$ is 7, then $\mathbf{H}$ is not an O-sequence, and so it is not level by Theorem 2.3. Moreover, by a simple calculation using CoCoA [1], we know that $\mathbf{H}$ is not level if the socle degree d of $\mathbf{H}$ is $4,\ 5,\ 6$. Furthermore, any graded ring with Hilbert function $\mathbf{H}$ has a socle element in degree $d-2$.

The following algorithm is to obtain the minimal free resolution of the lex-segment ideal whose Hilbert function is the given generic Artinian O-sequence of codimension 3.

In other words, one can use this algorithm to obtain all possible h-vectors which are of type

$$H = 1 \binom{2+1}{1} \binom{2+2}{2} \cdots \binom{2+(d-1)}{(d-1)} \underbrace{h_d \ \cdots \ h_d}_{s-\text{times}} \ h_{d+s}$$

with h_{d+s}=$h_d + 1$, and their type vectors $\mathcal{T}$(see [2] for the definition of n-type vectors) and minimal free resolutions.

Algorithm 3.4.

```
Define ADDUPHilbert(L)
  S:=[];
    For I := 2 To Len(L)
      Do S1:=Sum(First(L,I));
      Append(S,S1);
    EndFor;
   S2:=Comp(S,Len(S));
   S:=[S];
   Append(S,S2);
   Return S;
EndDefine;
```

```
--Define FLAT(N)
N:=c;
     List:=[1,3];
     COUNT:=0;
       For I := 2 To N-1 Do
           MaxN := BinExp(Comp(List,I),Len(List)-1,1,1);
              If N <= MaxN Then
                 Append(List,N);
                 Append(List,N+1);
                 If Comp(List,I) = N Then
                   -- OSEQUENCE(List);
                      Print List, NewLine," -> ";
                      ADHList:=ADDUPHilbert(List);
                      Print ADHList, NewLine," -> ";
                      TV.FromHF(ADHList);
                      TV.PrintRes(It);
                      Print NewLine, NewLine;
                  EndIf;
                  Remove(List,Len(List));
               Else Append(List,MaxN);
               EndIf;
```

```
      EndFor;
--EndDefine;
```

Here is an example how to use Algorithm 3.4 from CoCoA [1].

Example 3.5. Using Algorithm 3.4 with c=15 in the middle of the algorithm, we obtain the output as follows.

```
[1, 3, 6, 10, 15, 15, 16]
 -> [[4, 10, 20, 35, 50, 66], 66]
 -> [[[1]], [[1],[2]], [[1],[3],[4],[5]], [[1],[2],[3],[4],
     [5],[6]], [[1],[2],[3],[4],[5],[6],[7]]]
 0 --> R^3(-7)(+)R(-8)(+)R^16(-9) --> R^8(-6)(+)R^3(-7)
    (+)R^34(-8)
   --> R^6(-5)(+)R^2(-6)(+)R^18(-7) --> R

[1, 3, 6, 10, 15, 15, 15, 16]
 -> [[4, 10, 20, 35, 50, 65, 81], 81]
 -> [[[1]], [[1],[2]], [[1],[2],[4],[6]], [[1],[2],[3],[4],
     [5],[6],[7]], [[1],[2],[3],[4],[5],[6],[7],[8]]]
 0 --> R^3(-7)(+)R^2(-8)(+)R(-9)(+)R^16(-10)
   --> R^8(-6)(+)R^5(-7)(+)R^2(-8)(+)R^34(-9)
```

```
   --> R^6(-5)(+)R^3(-6)(+)R(-7)(+)R^18(-8) --> R


[1, 3, 6, 10, 15, 15, 15, 15, 16]
 -> [[4, 10, 20, 35, 50, 65, 80, 96], 96]
 -> [[[1]], [[1],[2]], [[1],[2],[4],[5]], [[2],[3],[4],[5],
     [6],[7],[8]], [[1], [2], [3], [4], [5], [6], [7], [8],
     [9]]]
 0 --> R^3(-7)(+)R^2(-8)(+)R^2(-9)(+)R^16(-11)
   --> R^8(-6)(+)R^5(-7)(+)R^4(-8)(+)R(-9)(+)R^33(-10)
   --> R^6(-5)(+)R^3(-6)(+)R^2(-7)(+)R(-8)(+)R^17(-9)
   --> R
                          .
                          .
                          .

[1, 3, 6, 10, 15, 15, 15, 15, 15, 15, 15, 15, 15, 16]
 -> [[4, 10, 20, 35, 50, 65, 80, 95, 110, 125, 140, 155,
      171], 171]
 -> [[[1]], [[1],[2]], [[1],[2],[4],[5]], [[1],[3],[5],
     [7],[9], [12],[13]],[[1],[2],[3],[4],[5],[6],[7],[8],
     [9],[10],[11],[12],[13],[14]]]
 0 --> R^3(-7)(+)R^2(-8)(+)R^2(-9)(+)R(-10)(+)R(-11)
```

```
    (+)R(-12)(+)R(-13)(+)R(-14)(+)R^16(-16)
   --> R^8(-6)(+)R^5(-7)(+)R^4(-8)(+)R^3(-9)(+)R^2(-10)
    (+)R^2(-11)(+)R^2(-12)(+)R^2(-13)(+)R^33(-15)
   --> R^6(-5)(+)R^3(-6)(+)R^2(-7)(+)R^2(-8)(+)R(-9)
    (+)R(-10)(+)R(-11)(+)R(-12)(+)R^17(-14)
   --> R


[1, 3, 6, 10, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 16]
 -> [[4, 10, 20, 35, 50, 65, 80, 95, 110, 125, 140, 155,
      170, 186],186]
 -> [[[1]], [[1],[2]], [[1],[2],[4],[5]], [[1],[3],[5],[7],
     [9],[11],[14]],[[1],[2],[3],[4],[5],[6],[7],[8],[9],
     [10],[11],[12],[13],[14],[15]]]
 0 --> R^3(-7)(+)R^2(-8)(+)R^2(-9)(+)R(-10)(+)R(-11)
    (+)R(-12)(+)R(-13)(+)R(-14)(+)R(-15)(+)R^16(-17)
   --> R^8(-6)(+)R^5(-7)(+)R^4(-8)(+)R^3(-9)(+)R^2(-10)
    (+)R^2(-11)(+)R^2(-12)(+)R^2(-13)(+)R^2(-14)(+)R^33(-16)
   --> R^6(-5)(+)R^3(-6)(+)R^2(-7)(+)R^2(-8)(+)R(-9)
    (+)R(-10)(+)R(-11)(+)R(-12)(+)R(-13)(+)R^17(-15)
   --> R


[1, 3, 6, 10, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 16]
```

```
 -> [[4, 10, 20, 35, 50, 65, 80, 95, 110, 125, 140, 155, 170,
      185, 201], 201]
 -> [[[1]], [[1],[2]], [[1],[2],[4],[5]], [[1],[3],[5],[7],
     [9],[11],[13]], [[1],[2],[3],[4],[5],[6],[7],[8],[9],
     [10],[11],[12],[13],[14],[15],[16]]]
 0 --> R^3(-7)(+)R^2(-8)(+)R^2(-9)(+)R(-10)(+)R(-11)(+)R(-12)
    (+)R(-13)(+)R(-14)(+)R(-15)(+)R(-16)(+)R^16(-18)
   --> R^8(-6)(+)R^5(-7)(+)R^4(-8)(+)R^3(-9)(+)R^2(-10)
    (+)R^2(-11)(+)R^2(-12)(+)R^2(-13)(+)R^2(-14)(+)R^2(-15)
    (+)R^33(-17)
   --> R^6(-5)(+)R^3(-6)(+)R^2(-7)(+)R^2(-8)(+)R(-9)(+)R(-10)
    (+)R(-11)(+)R(-12)(+)R(-13)(+)R(-14)(+)R^17(-16)
   --> R
```

From the above output, we obtain 10 cases. However, we omit some of them because of too much space to put here.

Now we prove that the following interesting case among the above all 10 cases in Example 3.5 in Proposition 3.6

Proposition 3.6. *The Artinian O-sequence* $1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 15 \quad 16$ *is not level.*

Proof. Let I be an ideal in $R = k[x, y, z]$ with Hilbert function 1 3 6 10 15 15 16. Then the minimal free resolution of R/I^{lex} is

$$\begin{array}{lllll} 0 & \to & R^3(-7) \oplus R(-8) \oplus R^{16}(-9) & \to & R^8(-6) \oplus R^3(-7) \oplus R^{34}(-8) \\ & \to & R^6(-5) \oplus R^2(-6) \oplus R^{18}(-7) & \to & R \quad \to \quad R/I \quad \to \quad 0. \end{array}$$

Suppose that I has 18 generators in degree 7. Let J be the ideal generated by the components of I of degree ≤ 6. Then the Hilbert function of R/J is 1 3 6 10 15 15 16 18 $\cdots$. Note that the sequence 14 16 18 in degrees 5, 6 and 7 has a maximal growth. Therefore, by Theorem 2.3, R/J has a socle element in degree 5. Hence also R/I has such a socle element. Since R/I and R/J agree in drgree ≤ 6. It follows that in order for R/I to be level, I must have at most 17 generators in degree 7. But then both copies $R(-7)$ of the last free module cannot be canceled. □

By the same method as in the proof of Proposition 3.6, one can show that the following cases in Table 1 cannot be level.

1, 3, 6, 10, 10, 11	1, 3, 6, 10, 12, 12, 13
1, 3, 6, 10, 13, 13, 14	1, 3, 6, 10, 14, 14, 15
1, 3, 6, 10, 14, 14, 14, 15	1, 3, 6, 10, 15, 15, 15, 16
1, 3, 6, 10, 15, 16, 16, 17	1, 3, 6, 10, 15, 16, 16, 16, 17
1, 3, 6, 10, 15, 17, 17, 18	$\cdots$

TABLE 1

We use Algorithm 3.4 to obtain the generic cases of O-sequences, their vectors and minimal free resolutions.

Here is an algorithm for the general cases.

Algorithm 3.7.

```
Define ADDUPHilbert(L)
  S:=[];
    For I := 2 To Len(L)
      Do S1:=Sum(First(L,I));
      Append(S,S1);
    EndFor;
   S2:=Comp(S,Len(S));
   S:=[S];
   Append(S,S2);
   Return S;
EndDefine;

Define FindFlat(K)
  L := [[[1,3]]];
  For I:=2 To K Do
   Prov := [];
   Foreach El In L[I-1] Do
```

```
    For J:=Comp(El,Len(El)) To BinExp(El[Len(El)],I-1,1,1) Do
     If J < K + 2 And (J > Comp(El,Len(El)) Or J = K) Then
       Append(Prov,Concat(El,[J]));
       EndIf;
      EndFor;
    EndForeach;
    Append(L, Prov);
  EndFor;

FlatList:=[];
Foreach El In L Do
     A:=[Elm In El | Elm[Len(Elm)] = K+1
            And Elm[Len(Elm)-1] = K
            And Elm[Len(Elm)-2] = K];
     Foreach Ell In A Do
            Append(FlatList, Ell);
     EndForeach;
EndForeach;
Return FlatList;
EndDefine;

FlatList:=FindFlat(c);
```

```
Foreach El In FlatList Do
Print El, NewLine," -> ";
ADHList:=ADDUPHilbert(El);
Print ADHList, NewLine," -> ";
TV.FromHF(ADHList);
TV.PrintRes(It);
Print NewLine, NewLine;
EndForeach;
```

Now we give an example how to produce O-sequences using Algorithm 3.7 which cannot be constructed using the previous Algorithm 3.4 in general.

Example 3.8. Using Algorithm 3.7 with c=15 in the middle of the algorithm, we obtain the output as follows.

```
[1, 3, 6, 10, 15, 15, 16]
 -> [[4, 10, 20, 35, 50, 66], 66]
 -> [[[1]], [[1],[2]], [[1],[3],[4],[5]], [[1],[2],[3],[4],
     [5],[6]], [[1],[2],[3],[4],[5], [6],[7]]]
 0 --> R^3(-7)(+)R(-8)(+)R^16(-9) --> R^8(-6)(+)R^3(-7)(+)
       R^34(-8)
   --> R^6(-5)(+)R^2(-6)(+)R^18(-7) --> R
```

```
[1, 3, 6, 9, 12, 15, 15, 16]
 -> [[4, 10, 19, 31, 46, 61, 77], 77]
 -> [[[1],[2],[4],[6]], [[1],[2],[3],[4],[5],[6],[7]],
    [[1],[2],[3], [4],[5],[6],[7],[8]]]
 0 --> R^2(-8)(+)R(-9)(+)R^16(-10) --> R^5(-7)(+)
       R^2(-8)(+)R^34(-9)
   --> R(-3)(+)R^3(-6)(+)R(-7)(+)R^18(-8) --> R

[1, 3, 6, 10, 12, 15, 15, 16]
 -> [[4, 10, 20, 32, 47, 62, 78], 78]
 -> [[[1]], [[1],[2],[4],[6]], [[1],[2],[3],[4],[5],[6],
     [7]], [[1],[2],[3],[4],[5],[6],[7],[8]]]
 0 --> R(-6)(+)R^2(-8)(+)R(-9)(+)R^16(-10)
   --> R^3(-5)(+)R^5(-7)(+)R^2(-8)(+)R^34(-9)
   --> R^3(-4)(+)R^3(-6)(+)R(-7)(+)R^18(-8) --> R

[1, 3, 6, 10, 13, 15, 15, 16]
 -> [[4, 10, 20, 33, 48, 63, 79], 79]
 -> [[[2]], [[1],[2],[4],[6]], [[1],[2],[3],[4],[5],[6],
     [7]], [[1],[2],[3],[4], [5], [6], [7], [8]]]
 0 --> R(-7)(+)R^2(-8)(+)R(-9)(+)R^16(-10)
```

```
   --> R(-5)(+)R^2(-6)(+)R^5(-7)(+)R^2(-8)(+)R^34(-9)
   --> R^2(-4)(+)R(-5)(+)R^3(-6)(+)R(-7)(+)R^18(-8)
   --> R

[1, 3, 6, 10, 14, 15, 15, 16]
 -> [[4, 10, 20, 34, 49, 64, 80], 80]
 -> [[[1],[2]], [[1],[2],[4],[6]], [[1],[2],[3],[4],[5],
     [6],[7]], [[1],[2],[3],[4],[5],[6],[7],[8]]]
 0 --> R^2(-7)(+)R^2(-8)(+)R(-9)(+)R^16(-10)
   --> R^5(-6)(+)R^5(-7)(+)R^2(-8)(+)R^34(-9)
   --> R(-4)(+)R^3(-5)(+)R^3(-6)(+)R(-7)(+)R^18(-8)
   --> R

[1, 3, 6, 10, 15, 15, 15, 16]
 -> [[4, 10, 20, 35, 50, 65, 81], 81]
 -> [[[1]], [[1],[2]], [[1],[2],[4],[6]], [[1],[2],[3],
     [4],[5],[6],[7]],[[1], [2], [3], [4], [5], [6],
     [7], [8]]]
 0 --> R^3(-7)(+)R^2(-8)(+)R(-9)(+)R^16(-10)
   --> R^8(-6)(+)R^5(-7)(+)R^2(-8)(+)R^34(-9)
   --> R^6(-5)(+)R^3(-6)(+)R(-7)(+)R^18(-8) --> R
```

.

.

.

```
[1, 3, 6, 10, 13, 14, 15, 15, 15, 15, 15, 15, 15, 15, 15, 16]
 -> [[4,10,20,33,47,62,77,92,107,122,137,152,167,182,198],
     198]
 -> [[[2]], [[2],[4],[5]], [[1],[3],[5],[7],[9],[11],[13]],
    [[1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],
     [13],[14],[15],[16]]]
 0 --> R(-7)(+)R(-8)(+)R^2(-9)(+)R(-10)(+)R(-11)(+)R(-12)
    (+)R(-13)(+)R(-14)(+)R(-15)(+)R(-16)(+)R^16(-18)
   --> R(-5)(+)R^3(-6)(+)R^2(-7)(+)R^4(-8)(+)R^3(-9)
    (+)R^2(-10)(+)R^2(-11)(+)R^2(-12)(+)R^2(-13)
    (+)R^2(-14)(+)R^2(-15)(+)R^33(-17)
   --> R^2(-4)(+)R^2(-5)(+)R(-6)(+)R^2(-7)(+)R^2(-8)
    (+)R(-9)(+)R(-10)(+)R(-11)(+)R(-12)(+)R(-13)
    (+)R(-14)(+)R^17(-16)
   --> R
```

```
[1, 3, 6, 10, 13, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 16]
 -> [[4,10,20,33,48,63,78,93,108,123,138,153,168,183,199],
      199]
```

```
 -> [[[2]], [[1],[2],[4],[5]], [[1],[3],[5],[7],[9],[11],
     [13]], [[1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],
     [12],[13],[14],[15],[16]]]
 0 --> R(-7)(+)R^2(-8)(+)R^2(-9)(+)R(-10)(+)R(-11)(+)R(-12)
    (+)R(-13)(+)R(-14)(+)R(-15)(+)R(-16)(+)R^16(-18)
   --> R(-5)(+)R^2(-6)(+)R^5(-7)(+)R^4(-8)(+)R^3(-9)
    (+)R^2(-10)(+)R^2(-11)(+)R^2(-12)(+)R^2(-13)(+)R^2(-14)
    (+)R^2(-15)(+)R^33(-17)
   --> R^2(-4)(+)R(-5)(+)R^3(-6)(+)R^2(-7)(+)R^2(-8)(+)R(-9)
    (+)R(-10)(+)R(-11)(+)R(-12)(+)R(-13)(+)R(-14)(+)R^17(-16)
   --> R


[1, 3, 6, 10, 14, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 16]
 -> [[4,10,20,34,49,64,79,94,109,124,139,154,169,184,200],
      200]
 -> [[[1],[2]], [[1],[2],[4],[5]], [[1],[3],[5],[7],[9],[11],
     [13]], [[1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],
     [12],[13], [14],[15],[16]]]
 0 --> R^2(-7)(+)R^2(-8)(+)R^2(-9)(+)R(-10)(+)R(-11)(+)R(-12)
    (+)R(-13)(+)R(-14)(+)R(-15)(+)R(-16)(+)R^16(-18)
   --> R^5(-6)(+)R^5(-7)(+)R^4(-8)(+)R^3(-9)(+)R^2(-10)
    (+)R^2(-11)(+)R^2(-12)(+)R^2(-13)(+)R^2(-14)(+)R^2(-15)
```

```
    (+)R^33(-17)
   --> R(-4)(+)R^3(-5)(+)R^3(-6)(+)R^2(-7)(+)R^2(-8)(+)R(-9)
    (+)R(-10)(+)R(-11)(+)R(-12)(+)R(-13)(+)R(-14)(+)R^17(-16)
   --> R

[1, 3, 6, 10, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 16]
 -> [[4,10,20,35,50,65,80,95,110,125,140,155,170,185,201],
      201]
 -> [[[1]], [[1],[2]], [[1],[2],[4],[5]], [[1],[3],[5],[7],
     [9],[11], [13]], [[1],[2],[3],[4],[5],[6],[7],[8],[9],
     [10],[11],[12], [13],[14], [15], [16]]]
 0 --> R^3(-7)(+)R^2(-8)(+)R^2(-9)(+)R(-10)(+)R(-11)(+)R(-12)
    (+)R(-13)(+)R(-14)(+)R(-15)(+)R(-16)(+)R^16(-18)
   --> R^8(-6)(+)R^5(-7)(+)R^4(-8)(+)R^3(-9)(+)R^2(-10)
    (+)R^2(-11)(+)R^2(-12)(+)R^2(-13)(+)R^2(-14)(+)R^2(-15)
    (+)R^33(-17)
   --> R^6(-5)(+)R^3(-6)(+)R^2(-7)(+)R^2(-8)(+)R(-9)(+)R(-10)
    (+)R(-11)(+)R(-12)(+)R(-13)(+)R(-14)(+)R^17(-16)
   --> R
```

From the above output, we find all 321 cases. Recall we have 10 cases which are generic cases in Example 3.5.

Here is another interesting case which is not level.

Proposition 3.9. *The Artinian O-sequence* $\mathbf{H}: 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 20 \quad 20 \quad 20 \quad 21$ *cannot be level.*

Proof. Let I be an ideal in $R = k[x, y, z]$ with Hilbert function $1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 20 \quad 20 \quad 20 \quad 21$. Then the minimal free resolution of R/I^{lex} is

$$\begin{array}{rl} 0 & \to \; R^3(-8) \oplus R^2(-9) \oplus R(-10) \oplus R^{21}(-11) \\ & \to \; R^8(-7) \oplus R^5(-8) \oplus R^2(-9) \oplus R^{44}(-10) \\ & \to \; R(-5) \oplus R^5(-6) \oplus R^3(-7) \oplus R(-8) \oplus R^{23}(-9) \\ & \to \; R \; \to \; R/I \; \to \; 0. \end{array}$$

Suppose that I has 23 generators in degree 9. Let J be the ideal generated by the components of I of degree ≤ 8. Then the Hilbert function of R/J is $1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 20 \quad 20 \quad 20 \quad 21 \quad 23 \quad \cdots$. Note that the sequence $19 \quad 21 \quad 23$ in degrees 7, 8 and 9 has a maximal growth. Therefore, by Theorem 2.3, R/J has a socle element in degree 7. Hence also R/I has such a socle element. Since R/I and R/J agree in drgree ≤ 8. It follows that in order for R/I to be level, I must have at most 22 generators in degree 9. But then both copies $R(-9)$ of the last free module cannot be canceled. $\square$

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Abstract

Some Algorithms for Artinian O-sequences

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This thesis studies whether some kind of Artinian O-sequences of codimension 3 are level or not which is increasing, flat and increasing (i.e. the Artinian O-sequences of h_0 h_1 ... h_{d-1} h_d h_{d+1} with $h_{d-1} = h_d < h_{d+1}$).

In this thesis, two CoCoA algorithms which generate sequences for this research are implemented and described with some examples. Furthermore, based on minimal free resolution,

this thesis proves that the sequences generated by these two algorithms are non-level Artinian O-sequences.