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碩士學位 請求論文

Properties Fractals and application

프랙탈의 성질과 응용

2008 年

誠信女子大學校 教育大學院

教育學科 數學教育專攻

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認 准 書

金炤映의 碩士學位 論文을 認准함

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2008 年　月

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논문개요

우리 주변에서 흔히 관찰할 수 있는 브로콜리, 번개의 전파, 나뭇가지, 혈관 등은 작은 구조가 전체 구조와 비슷한 형태로 끝없이 되풀이 되는 프랙탈 구조를 갖는 것으로 이들 구조의 특징은 '프랙탈 차원을 갖는다'는 것과 '자기 닮음'이라는 두가지 성질에 있다. 특히 프랙탈 도형의 차원은 유클리드 기하도형의 차원과는 다른 의미를 갖는 것으로 얼마나 많이 구부려 있는가의 정도에 따라 1차원과 2차원 사이의 어느 것으로 결정된다.

이러한 특징은 작은 크기로 반복하거나 더 복잡한 구조를 나타내는 망막 세포나 영토의 범위등를 관찰하고 분석하는 등 일반적인 분야에 중요한 도구가 될 수 있다.

이 연구 논문에서는 일반적인 기하도형의 비율을 다르게 하여 프랙탈 구조를 갖는 도형의 정확한 차원을 구하는 방법이 유도되는 과정을 알아보겠다. 또 이를 이용하여 자연 현상에서 나타나는 프랙탈 구조를 수학적 모델로 세우고 프랙탈 이론의 성질을 응용하여 생물학적 현상의 차원을 구하는 방법을 제시해보겠다.

차 례

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Abstract

I Introduction

The problem with a good name for a new field, particularly one such as fractal theory which can be visually dramatic and practised without much background and sophistication, is that uniformed proselytising and inappropriate use can raise unrealistic expectations as to its relevance and applicability. Chaos and fractal theory look like proposed by some as biological panaceas fortunately, but there are enough realists to counter this view.

Although fractal theory taken to be a biological explanation of how these structures and patterns are formed which like many natural shapes such as trees, weeds, flowers, butterfly wing patterns and so on, they say essentially nothing about the biological processes and mechanisms which are involved in their development.

One of the applications of fractal theory is directly related to the measurement of biological structures at different magnification. We can think of fractals in a simplistic, but still useful, way as geometric figures which repeat themselves at progressively smaller scales or exhibit progressively more complex structure when observed at lager and lager magnifications. With a fractal there is often self-symmetry, or approximate self-symmetry. That is, if we magnify a small part of the overall pattern it more or less displays some aspects of the whole pattern.

we discuss some simple examples of fractals and how to generate them to highlight some of their essential properties to get the basic ideas of what a fractal and a fractal dimension are.

Biological Examples

Tauchi and Masland(1984) and Tauchi et al.(1992) studied the shape and arrangement of specific neuronal cells in the rabbit retina. Montague and Friedlander(1991) studied the morphogenesis and spatial coverage of isolated retinal ganglion cells using cat retinal cells. As a result, they were faced with qualitatively similar structures.

Surface and Volume Measurements of specific membranes in the rat hepatocyte have resulted in widely different values. Paumgartner et al. (1981)

concentrated on the effect of different magnification, or scales, at which the measurements were carried out.

Pulmonary Blood Flow. Pulmonary blood flow is an important field and has been the focus of a major long term study by Robertson, Glenny and their colleagues. Glenny and Robertson (1990; see also their general review article, 1991a, on applying fractal analysis to physiology) studied the heterogeneity of pulmonary blood flow using an analytic fractal procedure which they compared with the traditional gravity model. Importantly they compared their data with their fractal model and obtained a very good fit. Glenny and Robertson(1991b) compared two branching fractal models, one in which the branching ratio (that is, the fraction γ and $1-\gamma$ of blood going from the parent to two daughters was fixed and in the other where it could vary about a mean of 0.5) and found that both compared very well with the data. They clearly demonstrated that, not only dose gravitation play a secondary role in producing heterogeneity, but that fractal models provide a quantitative mechanism for describing and function of the pulmonary vascular tree. Glenny and Robertson(1995) simulated a three dimensional branching model for pulmonary perfusion. Spatial and temporal heterogeneous pulmonary perfusion is physiologically also important and has been investigated by Glenny et al.(1995, 1997). With the plethora of articles on the fractal character of a wide spectrum of phenomena in Nature, many with scant connection to reality far less experiments, the importance, both pedagogically and scientifically, of the Glenny and Robertson work on blood perfusion is that their modeling is firmly rooted in reality and this, plus the fact that it is so closely tied to their experimental work, makes their physiological conclusion seminal.

Fractal analysis has been applied successfully to a wide variety of natural phenomena and, it has proved to be a useful tool. There is no doubt that fractal analysis can be an important tool in this general area although in many instances, it has been overdone and inappropriately applied.

II Preliminaries

We start by considering a specific fractal called the von Koch curve first described in 1906. As is often the case, this fractal is recursively generated. We start with a line L_0 and replace the inner third of it with two equal line segments to form L_1 as in Figure 1. Then, with each straight line segment in L_1 do the same again to get L_2, then L_3 and so on. The limiting curve L_n as $n \to \infty$ is the fractal known as the von Koch curve. Such recursive procedures can generate curves and structures with some interesting properties. Denoting the lengths of L_1, L_2, $\ldots$ by s_0, s_1, $\ldots$ we see that $s_1 = (4/3)s_0$, $s_2 = (4/3)^2 s_0, \ldots$, $s_n = (4/3)^n s_0, \ldots$. That is, the length of each L increases at each iteration of the rule and the length $s_n \to \infty$ as $n \to \infty$; in other words, the limiting curve is of infinite length. Not only that, there is an infinite distance between any two points on the von Koch curve. From a practical point of view, and in anticipation of biological application, such a limit is unobtainable. However, what is relevant is that the length of the curves depends on the scale we are able to measure them. We discuss this in more detail below.

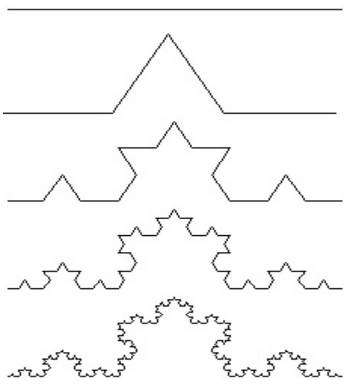


Figure 1: Construction of the von Koch curve. Here we divided the initial line L_0 into three equal segments and replace the middle one with two equal lines as in L_1. We then take each of the segments in L_1 and replace the middle third with two equal lines to get L_2. Doing this an infinite number of times results in the von Koch curve. The length, s, of each line, L, is $4/3$ longer than the previous one from which it was derived.

There is an obvious self-symmetry between subunits at one generating stage and the whole structure at a previous one, or even just a part of it. This is clear if we isolate some specimen sections and compare them with earlier L_n as shown in Figure 1. So, the von Koch curve, K, is self-similar and it has structures at however small a scale is we look at it. The self-similarity of subunits of the pattern at ever smaller scales is a particularly common property of certain fractals. In fact, a figure which exhibits this self-similarity at ever smaller scales is fractal although all fractals do not necessarily have this property; see Figure 2.

Let the unit of scale at any generation stage n be μ_n. Then, if we take the length of the line L_0 to be unity then $\mu_0 = 1$, $\mu_1 = 1/3$, $\mu_2 = (1/3)^2$,. . . ,$\mu_n = (1/3)^n$ and so on. We can relate the unit of scale, μ, and the magnification, M say, in a simple way. If we consider the structural subunit enclosed in the box in L_4 and magnify it by 3 we get L_3, while if we magnify the structural unit in the box in L_3 by 3^2 we get L_1: scale μ and the magnification M are related by $\mu \propto \lambda/M$ where λ is related to the resolvable scale unit at magnification unity. By this we mean that if we are looking at a magnified micrograph it is the unit in the image plane reflecting the test system under investigation at the smallest magnification. The von Koch fractal curve involves the magnification, M, which, from a biological application point of view, is not possible; we come back to this point below since it relates to how we calculate the 'dimension' of a fractal.

Another classical self-similar fractal is the Sierpinski triangle first the described in 1916. Here you start with an isosceles triangle and repeatedly

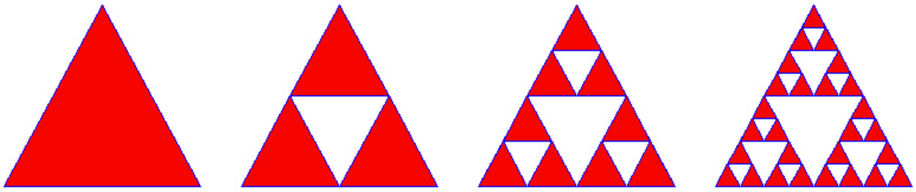


Figure 2: Sierpinski triangle fractal. The construction algorithm is to start with a triangle, remove the small inner triangle as in the second figure of four equal triangles and continue in this way with each remaining black triangle but, on a successively reduced scale .

remove similar triangles a quarter of the area. The initial stages are shown in Figure 2. As before we can calculate the unit of scale (an area here) at any stage. This fractal, as with the von Koch curve, can also be used to generate a variety of other fractal curves by using variations in the rule (see, for example, Peitgen et al. 1992). The existence of self-similar structure at any scale, however small, is clear.

We can now see how to construct self-similar fractals of whatever complexity you want. Self-similar fractals, however, are only one small class of fractal shapes. They involve linear scaling laws, which we discuss in more detail below. We can now see how one could, with a little ingenuity, devise specific fractal generators to produce a vast variety of shapes which can be tuned to look like all kinds of growing things such as tree, weeds, starfish, flowers, ferns, snowflakes, cauliflowers and ganglion cells. If we have nonlinearity in the fractal generators as well, the complex figures we can generate are unlimited. One example of such nonlinear fractals is Julia set, after Gaston Julia, who published his highly original and seminal study in 1918 when he was 25 years old. They are just one class of nonlinear fractals but possibly the best known since the resulting patterns can be very subtle and beautiful and the dramatic evolution of them can be easily displayed on a very basic personal computer with simple programmes. These sets involve transformations in the complex plane; see, for example, the discussion and numerous examples in the book by Peitgen et al.(1992).

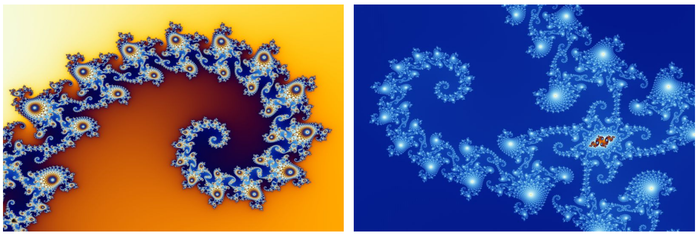


Figure 3: Example of a Julia set which exhibits fractal properties. The fractal generator is nonlinear. Note the similarity of the two successively enlarged parts. It is also possible to see in the third figure how the whole process continues, although without exact self-symmetry.

The Julia sets, which are nonlinear and hence not self-similar, are based on interactions of polynomials like $z_2 + k$, $z_2 + z + k$, $z^3 + k$ and so on, where z and k are complex numbers, z, as a pair of real numbers which determine a point in the plane. The generation algorithm involves repeated applications of the polynomial transformation. For example, start with the point z_0 and evaluate the point, $z_1 = z_0^2 + k$, then $z_2 = z_1^2 + k$ and so on. The sequence of points is a Julia set; Figure 3 is one computer-generated example. The study of these is interesting but it should be kept in mind that their connection with real biological applications is still moot. Later we briefly discuss the relevance of fractals to biological situations in general.

III Main Results

Consider a square of side S. If we scale down each side by a scale factor $r = 1/2$ then we need $4(= 2^2)$ of the smaller boxes to fill the original square. If we scale-down by a factor $r = 1/3$ we need 3^2 of the scaled-down squares to fill the original square. Generally if we scale down by a factor, r, we need $(1/r)^2$ reduced squares to fill the original square. If we do the same with a cube then, with a scale factor r, the number of cubes needed to fill the original box is $(1/r)^3$. The power of $1/r$ is directly related to the geometric dimension of the original figure, 2 for the square and 3 for the cube. If r is the scale factor and $m(r)$, which is a function of r, denotes the number of scaled-down pieces(similar to the original figure) which are needed to fill the original figure then for the square and the cube $m = (1/r)^D$, where D is the dimension, 2 for the square and 3 for the cube. This suggests a reasonable definition of the dimension, D, of a self-similar fractal, such as we derived above is given by

$$m = \left(\frac{1}{r}\right)^D \quad \Rightarrow \quad D = \frac{\ln m(r)}{\ln(1/r)} \tag{1}$$

on taking the logarithm of both sides. Strictly the fractal dimension, $D_{fractal}$, is given by the limit of this expression as the scale factor $r \to \infty$; that is, the actual unit of length scale tends to zero, and so

$$D_{fractal} = \lim_{r\to\infty} \frac{\ln m(r)}{\ln(1/r)}. \tag{2}$$

In the case of the von Koch curve the successive generations of the curve L_k, for any k, with length s_k, are made up of four equal pieces each of which is similar to the previous curve $L_k - 1$, with length $s_k - 1$, but scaled down by a factor of three. So, here the number of copies $m = 4$ and the scale factor $r = 1/3$. With the fractal definition of D in (2) we get the fractal dimension of the von Koch curve to be $D =$ ln 4/ln 3 $= 1.262$, which is indeed between 1 and 2.

Let us now consider a general scaling law and some quantity, Q say, which depends on the scale, r, at which we measure it. As an example, and to be specific, we can see that the finer the scale the more detail can be accounted for and, as a consequence, the measured length increases the finer the scale, r, that is, the larger the the magnification, we use. So, $Q(r)$ is a function of r and $Q = N(r)r$, where $N(r)$ is the number of scale units needed to cover the curve at scale r.

If a figure is self-similar we can say something about the behaviour of $N(r)$ as a function of r. Recall the discussion above with the square and cube and their subdivision into smaller squares and cubes. With the square when the scale factor was $r = 1/2$ we needed $N(r) = (1/r)^2$ of the smaller units to fill the original square. With the cube it was $N(r) = (1/r)^3$ of the smaller subunits. The powers of $1/r$ directly relate to the dimension of the quantity considered. So, in general for a self-similar scale law we have

$$N(r) = \frac{C}{r^D}, \tag{3}$$

where C is some constant and D is the dimension which characterises the quantity, Q. In the square and cube examples if we start with a known area and volume, C is that area and volume respectively. When dealing with a quantity we do not know, C is unknown a priori. With the expression for $N(r)$ in (3) we have

$$Q(r) = N(r)r = \frac{C}{r^{D-1}}. \tag{4}$$

Taking the logarithm of both sides gives

$$\ln Q(r) = -(D-1)\ln r + \ln C. \tag{5}$$

From a practical point of view, we plot $\ln Q(r)$ against $\ln r$ for various r and the gradient then determines the fractal dimension D.

The definition of the dimension in (4) is consistent with that given in (1). To see this, replace $Q(r)$ in (5) with the expression from (4) to get

$$\ln N(r) = -D\ln r + \ln r + \ln C$$

$$\Rightarrow D = \frac{\ln N(r)}{\ln(1/r)} + \frac{\ln C}{\ln(1/r)} \approx \frac{\ln N(r)}{\ln(1/r)} \quad \text{for } r \text{ small.} \tag{6}$$

This is the same as the definition of the dimension D defined in (1) since $N(r) = m(r)$. We should remember that it is magnification, M, which is used in experimental measurements rather than the unit of scale, r. There is a direct proportionality between M and r, namely, $M = \mu/r$ where μ is some proportionality factor which can be calculated. In terms of the magnification, M, the dimension D is then given by (5) which become

$$\ln Q(M) = -(D-1)\ln(1/M) + \ln[\mu^{-(D-1)}C]. \tag{7}$$

This last term in (7) is simply a constant and not important at this stage; we briefly talk about it later. If the quantity Q is fractal then the expression in (6) is a straight line in the log-log graph. Use of experimental measurements at different scales can suggest whether or not a biological quantity is fractal-or approximately so. A specific experimental example is the electron micrograph measurements by Paumgartner et al.(1981) of the inner and outer mitochondrial membranes and endoplasmic reticulum briefly mentioned above. Here Q represents length and surface measurements. They plotted the logarithm of the various measurements, Q, against the logarithm of $1/M$, where M is the magnification.

Non Self-Similar Fractals and the Box-Counting Dimension

If a fractal is not self-similar we have to generalise our concept of a fractal dimension. We also have to have some method for calculating it in any experimental situation. An example of a fractal which is not self-similar is shown in Figure 2. One method which is widely used is called box counting and which we now describe. There are others some of which are discussed in the books cited above. Basically all the methods rely on measurements of some quantity at different scales, r, or magnifications, M, in the above notation. Effectively the measurement at a given scale ignores irregularities at a smaller scale.

The box method involves covering the object of measurement with regular square boxes (circles or spheres are also used) of size r. If the measurements are in the plane then squares are used while cubes are used if the body measurements are three-dimensional. Let us suppose the original smooth curve has length L. Then the number of boxes $N(r)$ is related to the size of the box used which depends on the unit of scale r; here $N(r) \propto L/r$. If it is a region, with area A, then the number of boxes $N(r) \propto A/r^2$. In the case of a power law (3), $N(r) \propto L/r^D$ and the fractal box dimension is, from (6), then given as

$$D = \lim_{r \to 0} \frac{\ln N(r)}{\ln(1/r)} \tag{8}$$

as long as the limit exists.

If we now return to the cellular structures we can calculate the box dimension by covering the cell with a network of square boxes of given size (scale) which specifies r, and simply counting them to get $N(r)$. Then, progressively decrease the box size and plot $\ln N(r)$ against $\ln(1/r)$ for a range of r which gives a straight line from which we can determine the gradient and hence D.

One problem with the box dimension is that it is not always easy to find the minimal cover. There are other more complex and more accurate methods for calculating the fractal dimension, one of the best of which is the Hausdorff dimension which uses sets of shapes with different sizes. Although mathematically it gives a more accurate value for the dimension it is very much harder to calculate. The generating rule consists of starting with a square divided into nine equal small squares. Then choose one at random and remove it from the figure to get the set S_1. The remaining eight squares are then divided into nine equal smaller squares and again one of the smaller squares in each box is selected at random and discarded to obtain S_2. The procedure is then repeated. This is a fractal structure with qualitatively similar structures at each scale reduction and there is a kind of power scale law in operation in that the individual surviving boxes certainly obey one. How do we calculate the box dimension of this fractal structure? If we take the length of the side of the original square to be unity then S_1 consists of $N = 8$ equal squares of side 1/3; that is, the scale $r = 1/3$. The set S_2 consists of $N = 8^2$ squares with scale $r = (1/3)^2$. At the nth generation we have a structure S_n with $N = 8^n$ squares with side of length $r = (1/3)^n$. From (8) we thus have the box dimension given by

$$D = \lim_{r \to \infty} \frac{\ln 8^n}{\ln 3^n} = \frac{\ln 8}{\ln 3} = \frac{3 \ln 2}{\ln 3} = 1.893.$$

IV Practical Applications

The fractal dimension of fractal-fern

Figure 4: Bracken fern

If we scale down each side by a scale factor $r = 1/2$ then we need $N = 3.4$ of the smaller boxes to fill the original square. At the nth generation we have a structure with $N = 3.4^n$ squares with side of length $r = (1/2)^n$. From (8) we thus have the box dimension given by

$$D = \lim_{r \to \infty} \frac{\ln 3.4^n}{\ln 2^n} = \frac{\ln 3.4}{\ln 2} = \frac{1.22}{0.69} = 1.77.$$

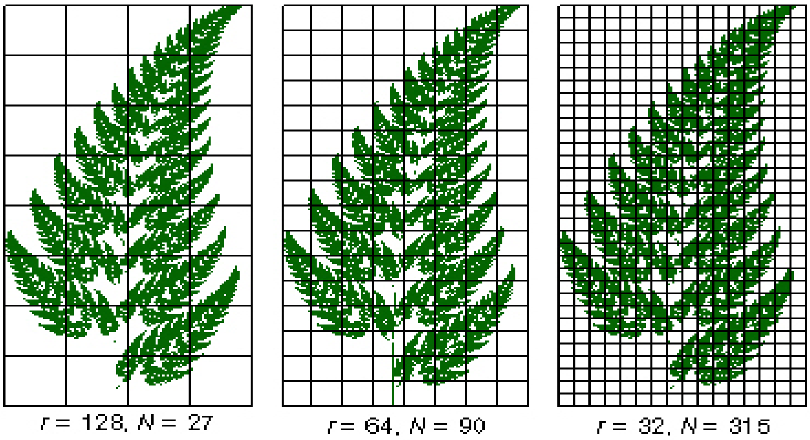


Figure 5: box dimension of fractal fern

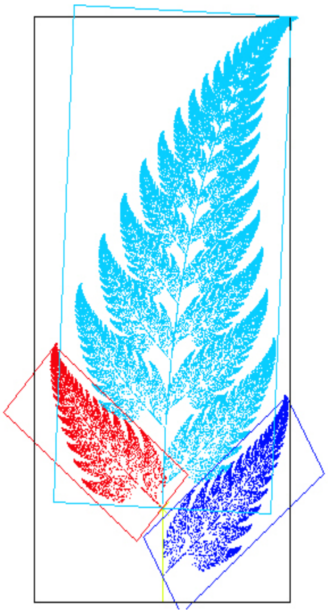
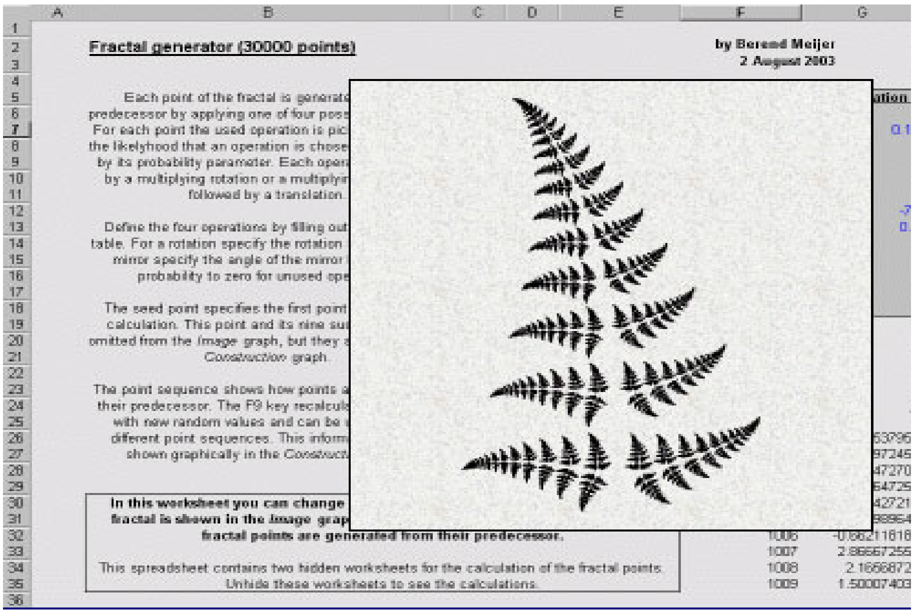


Figure 6: part of a fern leaf with the property of self-similar

Fractal fern generated by Iteration



Fractal generator (30000 points)

by Berend Meijer
2 August 2003

Each point of the fractal is generat
predecessor by applying one of four poss
For each point the used operation is pic
the likelyhood that an operation is chose
by its probability parameter. Each oper
by a multiplying rotation or a multiplyi
followed by a translation.

Define the four operations by filling out
table. For a rotation specify the rotation
mirror specify the angle of the mirror
probability to zero for unused ope

The seed point specifies the first point
calculation. This point and its nine su
omitted from the *Image* graph, but they
Construction graph.

The point sequence shows how points a
their predecessor. The F9 key recalcul
with new random values and can be
different point sequences. This inform
shown graphically in the *Construct*

In this worksheet you can change
fractal is shown in the *Image* grap
fractal points are generated from their predecessor.

This spreadsheet contains two hidden worksheets for the calculation of the fractal points.
Unhide these worksheets to see the calculations.

```
1006   -0.86211818
1007   2.86667255
1008   2.1666872
1009   1.50007403
```

Figure 7: Fractal Fern

The attractor of the iterated function system given by the set of *fern function* as in the following.

$$
\begin{aligned}
f_1(x,y) &= \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 1.60 \end{pmatrix} \\
f_2(x,y) &= \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.44 \end{pmatrix} \\
f_3(x,y) &= \begin{pmatrix} 0.20 & 0.24 \\ 0.23 & 0.22 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 1.60 \end{pmatrix} \\
f_4(x,y) &= \begin{pmatrix} 0.00 & 0.00 \\ 0.00 & 0.16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

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ABSTRACT

Properties Fractals and application

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We can observe fractal structure which shows a whole structure which repeat one part of oneself from a snowflake, broccoli and so on. This at a further a possibility becoming the important tool of the general field which is observes and analyzes scope of territory and the retina cell which there is repeats smaller size and more complex structure. In this thesis, with the mathematics model from the fractal structure which shows from natural phenomena and presents the method which buys the dimension of living thing apply which application of the fractal theory.