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석사학위 청구논문

Non-Level O-Sequences of codimension 6

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성신여자대학교 교육대학원

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이 논문을 석사학위논문으로 제출함.

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인준서

임지현의 석사학위 논문으로 인준함.

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논문개요

기존에 증명된 여차원이 4인 경우에 근거하여 여차원이 6일 때, level 이 될 가능성이 있는 Artinian O-수열에 대하여 level 여부를 연구하였다.

우선 $d < s$, $h_d \leq 2d+2$일 때, $h_{d-1} > h_d + 4$, $h_d = h_{d+1}$ 이고 H가 여차원이 6이면 H가 level 이 아님을 증명하였다.

그리고 여차원이 6인 몇 가지 Non-Unimodal Artinian O-수열이 level 이 될 수 없음을 보였다.

또한 Computer Program CoCoA 의 Algorithm을 소개하고, 그 Algorithm 을 이용하여 몇 가지 예들을 만들어 보았다.

마지막으로 Inverse System 을 도입하여 level 이 될 수 있는 예들을 만들어 보았다.

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Abstract

1. INTRODUCTION

Let $R = k[x_0, \ldots, x_n]$ be polynomial ring over an infinite field k of characteristic 0, and I be a homogeneous ideal of R. Then we have $R = \bigoplus_{i \geq 0} R_i$, where R_i is the vector space of dimension $\binom{i+n}{n}$ generated by the monomials in R having degree i. Since we can write $I := \bigoplus_{i \geq 0} I_i$, we get a graded ring

$$A = R/I := \bigoplus_{i \geq 0} (R_i/I_i) = \bigoplus_{i \geq 0} A_i$$

The ***Hilbert function*** of A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

In this thesis, we will study Artinian quotients $A = R/I$ of R where I is a homogeneous ideal of R. The ***h-vector*** of A is

$$\mathbf{H} := (h_0, h_1, \ldots, h_s),$$

where $h_i = \dim_k A_i$, and the last index s with $\dim_k A_s \neq 0$ is called the ***socle degree*** of A. The ***socle*** of A is defined by the annihilator of the maximal homogeneous ideal, namely

$$\operatorname{ann}_A(m) := \{a \in A \mid am = 0\} \qquad \text{where} \qquad m = \sum_{i=1}^{s} A_i.$$

Let h and i be positive integers. Then h can be written uniquely in the form

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expansion for h is called the ***i-binomial expansion*** of h. Also, define

$$h^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_{i-1} + 1}{(i-1)+1} + \cdots + \binom{m_j + 1}{j+1},$$

and $0^{\langle i\rangle} = 0$. Let $\mathbf{T} = (h_0, h_1, \dots, h_i, \dots)$ be a sequence of non-negative integers. We say that $\mathbf{T}$ is an ***O-Sequence*** if $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i\rangle}$ for all $i \geq 1$. Given an O-sequence $\mathbf{T} = (h_0, h_1, \dots, h_i, \dots)$, we can differentiate it to get a new sequence

$$\Delta\mathbf{T} = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \dots),$$

and we call $\Delta\mathbf{T}$ the ***first difference*** of $\mathbf{T}$. If $\Delta\mathbf{T}$ is an O-sequence again, $\mathbf{T}$ is called a ***differentiable O-Sequence***

Let $A = R/I$ be a Cohen-Macaulay ring of dimension d and let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to R \to R/I \to 0$$

be a minimal free resolution of R/I. Then A is a ***level algebra of m*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. In particular, if $m = 1$, then we say that A is a ***Gorenstein algebra*** and the h-vector of A is a ***Gorenstein sequence***. It is well-known that an Artinian graded algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_\ell$ is level if $\operatorname{soc}(A) = A_\ell$.

A total order on the set of the monomials of each degree is said to be a ***term order*** if $x_1 > \cdots > x_n$, and $m_1 \geq m_2$ implies $mm_1 \geq mm_2$ for any monomials m, m_1 and m_2.

The ***reverse lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if $\sum i_t > \sum j_t$ or $\sum i_t = \sum j_t$ and there is s such that $i_t = j_t$ for $1 \leq s < t$ and $i_s < j_s$.

The ***lexicographic order*** is a term order defined to be $x_1^{i_1}\cdots x_n^{i_n} > x_1^{j_1}\cdots x_n^{j_n}$ if and only if $\sum i_t > \sum j_t$ or $\sum i_t = \sum j_t$ and there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.

Let S be a subset of the set of the monomials in R_d. S is ***lex-segment*** if a monomial m of deree d is in S, then every monomial m' of degree d in R_d satisfying $m' > m$ with respect to the lexicographic order is in S.

Let $I = \oplus_{t\geq 0} I_t$ be a homogeneous ideal of R. We say that I is a ***lex$-$ segment ideal*** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment subset.

A monomial ideal I in R is ***stable*** if the monomial

$$\frac{x_j w}{x_{m(w)}}$$

belongs to I for every monomial $w \in I$ and $j < m(w)$ where

$$m(u) := \max\{j \mid a_j > 0\}$$

for $u = x_1^{a_1}\cdots x_n^{a_n}$.

Let S be a subset of all monomials in $R = \bigoplus_{i\geq 0} R_i$ of degree i. We call S a ***Borel fixed set*** if

$$u = x_1^{a_1}\cdots x_n^{a_n} \in S,\ a_j > 0 \quad \text{implies} \quad \frac{x_i u}{x_j} \in S$$

for every $1 \leq i \leq j \leq n$.

A monomial ideal I of R is called a ***Borel fixed ideal*** or ***strongly stable ideal*** if the set of all monomials in I_i is a Borel set for every

i. There are two Borel fixed monomial ideals canonically attached to a homogeneous ideal I of R: the generic initial ideal $\mathrm{Gin}(I)$ with respect to the reverse lex order and the lex-segment ideal I^{lex} associated to the ideal I. The ideal I^{lex} is defined as follows. For the vector space I_d of forms of degree d in I, one defines $(I^{\mathrm{lex}})_d$ to be the vector space generated by largest, in the lexicographical order, $\dim_k(I_d)$ monomials of degree d. By construction, I^{lex} is a strongly stable ideal and it only depends on the Hilbert function of I.

In [1], it has been proved that $\mathbf{H}$ is not level if $h_{d-1} > h_d = h_{d+1}$ for some $d < s$, $h_d \leq 2d+3$, and $\mathbf{H}$ is codimension 3. In this thesis, we show that $\mathbf{H}$ is not level, when $\mathbf{H}$ is of codimension 6 such that $h_{d-1} - h_d \geq 5$ and $h_d \geq 2d+2$ (see Theorem 2.6). We also prove that a certain non-unimodal O-sequence of codimension 6 cannot be level using Theorem 2.6

2. Some Non-Level O-sequences of Codimension 6

In [4], they studied the following interesting O-sequence:

$$\mathbf{H} \; : \; h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_{d+1} \quad \cdots \tag{1}$$

where $h_{d-1} > h_d = h_{d+1}$. Hence we have the following interesting question.

Question 2.1. Let $\mathbf{H}$ be as above. Is $\mathbf{H}$ NOT level?

In this section, we will prove that some O-sequence of codimension 6 is not level.

In [4], it has been proved that $\mathbf{H}$ is not level if $h_d \leq d+1$ (see the following proposition).

Proposition 2.2 (Proposition 2.21, [4]). *Let $h = (1, h_1, h_2, \ldots, h_s)$ be a h-vector of an Artinian algebra with socle degree s. Then h is **not** a level sequence if $h_d = h_{d+1} \leq d+1$ and $h_{d-1} > h_d$.*

We will revise the above proposition to the case in codimension 6 with $h_{d-1} > h_d + 4$ and $h_d \leq 2d+2$.

Let $R = k[x_1, \ldots, x_n]$ and $A = R/I$ where I is a homogeneous ideal of R having height n. Then A has the minimal free resolution $\mathcal{F}$, as an R-module, of the form:

$$0 \rightarrow \mathcal{F}_{n-1} \rightarrow \cdots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow R \rightarrow A \rightarrow 0$$

where $\mathcal{F}_j = \bigoplus_{t=1}^{\gamma_j} R^{\beta_{j,j+1+t}}(-(j+1+t))$ are each free graded R-modules.

The number $\{\beta_{j,i}\}$, for fixed j, $0 \leq j \leq n-1$, are called the j^{th} ***graded Betti numbers*** of I. It is well-known that the socle vector of A, $s(A) = (a_1, \ldots, a_s)$, is related to the $(n-1)^{st}$ graded Betti numbers as follows:

$$\beta_{n-1,n+i} = a_i.$$

It follows that A is a level algebra if and only if $\beta_{n-1,n+i} = 0$ for all $i \neq s$.

For I as above, the ***Betti diagram*** of R/I is a useful device to encode the graded Betti numbers of R/I (and hence of I). It is constructed as follows:

$$\begin{array}{c|cccccc}
 & & 0 & 1 & \dots & n-1 \\
\hline
0 & 1 & 0 & 0 & \dots & 0 \\
1 & 0 & * & * & \dots & * \\
t & 0 & \beta_{0,t+1} & \beta_{1,t+2} & * & \beta_{n-1,t+n} \\
d-2 & 0 & \beta_{0,d-1} & \beta_{1,d} & * & \beta_{n-1,d-2+n} \\
d-1 & 0 & \beta_{0,d} & \beta_{1,d+1} & * & \beta_{n-1,d-1+n} \\
d & 0 & \beta_{0,d+1} & \beta_{1,d+2} & * & \beta_{n-1,d+n}
\end{array}$$

Eliahou and Kervaire [2] studied minimal free resolutions of certain monomial ideals. We recall some of their results now.

Theorem 2.3 (Eliahou–Kervaire, [2]). *Let I be a stable monomial ideal of R (e.g., a lex segment ideal). Denote by $\mathcal{G}(I)_d$ the set of minimal generators of I which have degree d. Then*

$$\beta_{q,i} = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T)-1}{q}.$$

This theorem gives all the graded Betti numbers of the lex segment ideal just from an intimate knowledge of the generators of that ideal. Since the minimal free resolution of the ideal of a k-configuration in $\mathbb{P}^n$ is extremal ([5],[6]), we may apply this result to those ideals. It is an immediate consequence of the Eliahou–Kervaire Theorem that if I is either a lex-segment ideal or the ideal of a k-configuration in $\mathbb{P}^n$ which has *no* generators of degree d, then $\beta_{q,i} = 0$ whenever $i - q = d$.

By the result of [8], the only way we can cancel graded Betti numbers is if there are the same graded Betti numbers in the adjacent free modules of the extremal minimal free resolution. Note that it is quite obvious for a case of $n = 3$.

By the same idea as the proof of Theorem 4.4 in [1], we obtain the following lemma and proposition.

Lemma 2.4. *Let I be the lex-segment ideal in $R = k[x_1, x_2, x_3, x_4, x_5, x_6]$ with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_s)$ where $h_d = d + i$ and $1 \leq i \leq \frac{d^2+d}{2}$. Then the last monomial of I_d is*

$$\begin{array}{rcrcccl}
x_4 x_5^{i-1} x_6^{d-i}, & \textit{for} & 1 & \leq & i & \leq & d, \\
x_4^2 x_5^{i-(d+1)} x_6^{(2d-1)-i}, & \textit{for} & d+1 & \leq & i & \leq & 2d-1, \\
\vdots & & & & & & \\
x_4^{d-1} x_5^{i-\frac{d^2+d-4}{2}} x_6^{\frac{d^2+d-2}{2}-i}, & \textit{for} & \frac{d^2+d-4}{2} & \leq & i & \leq & \frac{d^2+d-2}{2}, \\
x_4^d, & \textit{for} & & & i & = & \frac{d^2+d}{2}.
\end{array}$$

Proposition 2.5. *Let $R = k[x_1, x_2, x_3, x_4, x_5, x_6]$ and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the h-vector of an Artinian algebra with socle degree s and*

$$h_d = h_{d+1} = d + i, \quad h_{d-1} > h_d, \quad \textit{and} \quad j := h_{d-1} - h_d$$

for $i = 1, 2, \ldots, \frac{d^2+d}{2}$*. Then, for every* $1 \leq k \leq d$ *and* $0 \leq \ell \leq d$*,*

$$\beta_{4,d+5} = \begin{cases} 5k-4, & \textit{for } (k-1)d - \frac{k(k-3)}{2} \leq i \leq (k-1)d - \frac{k(k-3)}{2} + (k-1), \\ 5k, & \textit{for } (k-1)d - \frac{k(k-3)}{2} + k \leq i \leq kd - \frac{(k-1)k}{2}. \end{cases}$$

$$\beta_{5,d+5} = j + \ell, \qquad \textit{for } (\ell-1)d - \frac{(\ell-2)(\ell-1)}{2} < i \leq \ell d - \frac{(\ell-1)\ell}{2}.$$

Proof. Since we assume $h_d = d + i$, the monomials not in I_d are the last $d + i$ monomials of R_d. By Lemma 2.4, the last monomial of $R_1 I_d$ is

$$\begin{array}{ccl} x_4 x_5^{i-1} x_6^{d-i+1}, & \text{for} & i = 1, \ldots, d, \\ x_4^2 x_5^{i-(d+1)} x_6^{2d-i}, & \text{for} & i = d+1, \ldots, 2d-1, \\ \vdots & & \\ x_4^{d-1} x_5^{i - \frac{d^2+d-4}{2}} x_6^{\frac{d^2+d}{2} - i}, & \text{for} & i = \frac{d^2+d-4}{2}, \frac{d^2+d-2}{2}, \\ x_4^d x_6, & \text{for} & i = \frac{d^2+d}{2}. \end{array}$$

In what follows, the first monomial of $I_{d+1} - R_1 I_d$ is

$$\begin{array}{ccl} x_5^{d+1}, & \text{for} & i = 1, \\ x_4 x_5^{i-2} x_6^{(d+2)-i}, & \text{for} & i = 2, \ldots, d, \\ \vdots & & \\ x_4^{d-1} x_5 x_6, & \text{for} & i = \frac{d^2+d-2}{2}, \\ x_4^{d-1} x_5^2, & \text{for} & i = \frac{d^2+d}{2}. \end{array} \tag{2}$$

Note that

$$\begin{array}{rcl} (d+i)^{\langle d \rangle} & = & (d+i) + k, \quad \text{for } i = (k-1)d - \frac{k(k-3)}{2}, \ldots, kd - \frac{k(k-1)}{2}, \\ & & \text{and } k = 1, \ldots, d. \end{array} \tag{3}$$

We now calculate the Betti number

$$\beta_{4,d+5} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T) - 1}{4}.$$

Based on equation (2), we will find this Betti number of two cases for i as follows.

Case 1-1. $i = (k-1)d - \frac{k(k-3)}{2}$ and $k = 1, 2, \ldots, d$.

Then, by equation (3), I_{d+1} has k-generators, which are

$$x_4^{k-1}x_5^{(d+2)-k}, x_4^{k-1}x_5^{(d+1)-k}x_6, \ldots, x_4^{k-1}x_5^{(d+3)-2k}x_6^{k-1}.$$

By the similar argument, I_{d+1} has k-generators including the element $x_4^{k-1}x_5^{(d+2)-k}$ for $i = (k-1)d - \frac{k(k-3)}{2} + 1, \ldots, (k-1)d - \frac{k(k-3)}{2} + (k-1)$. Hence we have that

$$\beta_{4,d+5} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T)-1}{4} = 5 \times (k-1) + 1 = 5k - 4.$$

Case 1-2. $i = (k-1)d - \frac{k(k-3)}{2} + k = (k-1)d - \frac{k(k-5)}{2}, \ldots, kd - \frac{k(k-1)}{2}$ and $k = 1, 2, \ldots, d$.

Then, by equation (3), I_{d+1} has k-generators, which are

$$x_4^k x_5^{i - \ (k-1)d - \frac{k^2-3k-2}{2}} x_6^{kd - \frac{k^2-k-4}{2} - i}, \ldots, x_4^k x_5^{i - \left((k-1)d - \frac{k(k-5)}{2}\right)} x_6^{\left(kd - \frac{k(k-3)}{2} + 1\right) - i}.$$

Hence we have that

$$\beta_{4,d+5} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T)-1}{4} = 5 \times k = 5k.$$

Now we move on the Betti number:

$$\beta_{5,d+5} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T)-1}{5}.$$

Recall $h_d = d + i$ and $j := h_{d-1} - h_d$. The calculation in this case is much more complicated, and there are four cases based on i and j.

Case 2-1. $(\ell-1)d-\frac{(\ell-2)(\ell-1)}{2} < i < \ell d-\frac{(\ell-1)\ell}{2}$ and $\ell = 1, 2, \ldots, d$.

Then the last monomial of I_d is

$$x_4^{\ell}x_5^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}}x_6^{\ell d-\frac{(\ell-1)\ell}{2}-i}.$$

(a) $(k-1)d-\frac{(k-1)k}{2} < i+j < kd-\frac{k(k+1)}{2}$ and $k = \ell, \ell+1, \ldots, d$.

Then the first monomial of $I_d - R_1I_{d-1}$ is

$$x_4^kx_5^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)}x_6^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$\begin{array}{l}
x_4^kx_5^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)}x_6^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)}, \ldots, x_4^kx_6^{d-k}, \\
x_4^{(k-1)}x_5^{d-(k-1)}, x_4^{(k-1)}x_5^{(d-1)-(k-1)}x_6, \ldots, x_4^{(k-1)}x_6^{d-(k-1)}, \\
\qquad\qquad\vdots \\
x_4^{\ell+1}x_5^{(d-1)-\ell}, x_4^{\ell+1}x_5^{(d-2)-\ell}x_6, \ldots, x_4^{\ell+1}x_6^{(d-1)-\ell} \\
x_4^{\ell}x_5^{d-\ell}, \ldots, x_4^{\ell}x_5^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}}x_6^{\ell d-\frac{(\ell-1)\ell}{2}-i}
\end{array}$$

and thus

$$\beta_{5,d+5} = \sum_{T\in\mathcal{G}(I)_d}\binom{m(T)-1}{5} = j+\ell,$$

(b) $i+j = (k-1)d-\frac{(k-1)k}{2}$ and $k = \ell+1, \ldots, d$.

Then the first monomial of $I_d - R_1I_{d-1}$ is

$$x_4^{k-1}x_5^{d-(k-1)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$\begin{aligned}
&x_4^{k-1}x_5^{d-(k-1)}, x_4^{k-1}x_5^{(d-1)-(k-1)}x_6, \dots, x_4^{k-1}x_6^{d-(k-1)},\\
&\qquad\vdots\\
&x_4^{\ell+1}x_5^{(d-1)-\ell}, x_4^{\ell+1}x_5^{(d-2)-\ell}x_6, \dots, x_4^{\ell+1}x_6^{(d-1)-\ell}\\
&x_4^{\ell}x_5^{d-\ell}, \dots, x_4^{\ell}x_5^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}}x_6^{\ell d-\frac{(\ell-1)\ell}{2}-i}
\end{aligned}$$

and thus

$$\beta_{5,d+5} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{5} = j+\ell.$$

Case 2-2. $i = \ell d - \frac{(\ell-1)\ell}{2}$ and $\ell = 1, 2, \dots, d$.

Then the last monomial of I_d is

$$x_4^{\ell}x_5^{d-\ell}.$$

(a) $(k-1)d - \frac{(k-1)k}{2} < i+j < kd - \frac{k(k+1)}{2}$ and $k = \ell+1, \dots, d$.

Then the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_4^{k}x_5^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)}x_6^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$\begin{aligned}
&x_4^{k}x_5^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)}x_6^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)}, \dots, x_4^{k}x_6^{d-k},\\
&x_4^{(k-1)}x_5^{d-(k-1)}, x_4^{(k-1)}x_5^{(d-1)-(k-1)}x_6, \dots, x_4^{(k-1)}x_6^{d-(k-1)},\\
&\qquad\vdots\\
&x_4^{\ell+1}x_5^{(d-1)-\ell}, x_4^{\ell+1}x_5^{(d-2)-\ell}x_6, \dots, x_4^{\ell+1}x_6^{(d-1)-\ell}\\
&x_4^{\ell}x_5^{d-\ell},
\end{aligned}$$

and thus

$$\beta_{5,d+5} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{5} = j+\ell.$$

(b) $i+j=(k-1)d-\frac{(k-1)k}{2}$ and $k=\ell+1,\dots,d$.

Then the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_5^{(k-1)} x_6^{d-(k-1)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$\begin{array}{l} x_4^{(k-1)}x_5^{d-(k-1)}, x_4^{(k-1)}x_5^{(d-1)-(k-1)}x_6, \dots, x_4^{(k-1)}x_6^{d-(k-1)}, \\ \qquad\qquad \vdots \\ x_4^{\ell+1}x_5^{(d-1)-\ell}, x_4^{\ell+1}x_5^{(d-2)-\ell}x_6, \dots, x_4^{\ell+1}x_6^{(d-1)-\ell} \\ x_4^{\ell}x_5^{d-\ell}, \end{array}$$

and thus

$$\beta_{5,d+5} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{5} = j+\ell,$$

as we wished. □

Now we prove the main result here.

Theorem 2.6. *Let* $\mathbf{H}$ *and* j *be as in Proposition 2.5. Then for every* $i=-(d-1),\dots,-1,0,1,2,\dots,d,d+1,d+2,$ *and* $j\geq 5$, $\mathbf{H}$ *is not level.*

Proof. By Proposition 3.8 in [4], this proposition holds for $i=-(d-1),\dots,-1,0,1$. Hence we may assume that $2\leq i\leq d+2$.

Let J be a lex-segment ideal in $R=k[x_1,x_2,x_3,x_4,x_5,x_6]$ such that the Hilbert function R/J is $\mathbf{H}$.

Then, by Proposition 2.5, we have that

$$\begin{aligned} \beta_{4,d+5}(J) &= \begin{cases} 5, & \text{for} \quad i = 2, \dots, d, \\ 6, & \text{for} \quad i = d+1, d+2, \end{cases} \qquad \text{and} \\ \beta_{5,d+5}(J) &= \begin{cases} j+1, & \text{for} \quad i = 2, \dots, d, \\ j+2, & \text{for} \quad i = d+1, d+2. \end{cases} \end{aligned} \tag{4}$$

Hence if $j \geq 5$, then we have $\beta_{5,d+5} > \beta_{4,d+5}$. Therefore $\mathbf{H}$ is not level, as we wished. □

Example 2.7. Consider an Artinian O-sequence

$$\mathbf{H} \;:\; 1 \quad 6 \quad 20 \quad 40 \quad 50 \quad 19 \quad 14 \quad 14 \quad 0.$$

Then $h_d = h_6 = 2 \times 6 + 2 = 14$ and $h_{d-1} - h_d = h_5 - h_6 = 19 - 14 = 5$. Hence, by Theorem 2.6, $\mathbf{H}$ is not level.

Example 2.8. Consider an Artinian O-sequences

$$\begin{array}{lcccccccccc} \mathbf{G_1} & : & 1 & 6 & 20 & 40 & 50 & 18 & 14 & 14 & 0. \\ \mathbf{G_2} & : & 1 & 6 & 20 & 40 & 50 & 20 & 15 & 15 & 0. \end{array}$$

which do not satisfy the condition of Theorem 2.6.

Let I and J be lex-segment ideals in $R = k[x_1, x_2, x_3, x_4, x_5, x_6]$ such that the Hilbert functions of R/I and R/J are $\mathbf{G_1}$ and $\mathbf{G_2}$, respectively.

Then the minimal free resolutions of R/I and R/J are

$$\begin{array}{rll}
0 & \rightarrow & R^2(-8) \oplus R^{11}(-9) \oplus R^{36}(-10) \oplus \boldsymbol{R^6}(\mathbf{-11}) \oplus R(-12) \oplus R^{14}(-13) \\
 & \rightarrow & R^{12}(-7) \oplus R^{62}(-8) \oplus R^{197}(-9) \oplus R^{32}(-10) \oplus \boldsymbol{R^6}(\mathbf{-11}) \oplus R^{71}(-12) \\
 & \rightarrow & R^{31}(-6) \oplus R^{143}(-7) \oplus R^{434}(-8) \oplus R^{68}(-9) \oplus R^{14}(-10) \oplus R^{144}(-11) \\
 & \rightarrow & R^{43}(-5) \oplus R^{170}(-6) \oplus R^{481}(-7) \oplus R^{72}(-8) \oplus R^{16}(-9) \oplus R^{146}(-10) \\
 & \rightarrow & R^{32}(-4) \oplus R^{105}(-5) \oplus R^{268}(-6) \oplus R^{38}(-7) \oplus R^{9}(-8) \oplus R^{74}(-9) \\
 & \rightarrow & R(-2) \oplus R^{10}(-3) \oplus R^{27}(-4) \oplus R^{60}(-5) \oplus R^{8}(-6) \oplus R^{2}(-7) \oplus R^{15}(-8) \\
 & \rightarrow & R \ \rightarrow \ R/I \ \rightarrow \ 0, \qquad \text{and} \\
0 & \rightarrow & R^2(-8) \oplus R^{11}(-9) \oplus R^{35}(-10) \oplus \boldsymbol{R^7}(\mathbf{-11}) \oplus R^2(-12) \oplus R^{15}(-13) \\
 & \rightarrow & R^{12}(-7) \oplus R^{62}(-8) \oplus R^{191}(-9) \oplus R^{38}(-10) \oplus \boldsymbol{R^{10}}(\mathbf{-11}) \oplus R^{77}(-12) \\
 & \rightarrow & R^{31}(-6) \oplus R^{143}(-7) \oplus R^{420}(-8) \oplus R^{82}(-9) \oplus R^{20}(-10) \oplus R^{158}(-11) \\
 & \rightarrow & R^{43}(-5) \oplus R^{170}(-6) \oplus R^{465}(-7) \oplus R^{88}(-8) \oplus R^{20}(-9) \oplus R^{162}(-10) \\
 & \rightarrow & R^{32}(-4) \oplus R^{105}(-5) \oplus R^{259}(-6) \oplus R^{47}(-7) \oplus R^{10}(-8) \oplus R^{83}(-9) \\
 & \rightarrow & R(-2) \oplus R^{10}(-3) \oplus R^{27}(-4) \oplus R^{58}(-5) \oplus R^{10}(-6) \oplus R^{2}(-7) \oplus R^{17}(-8) \\
 & \rightarrow & R \ \rightarrow \ R/J \ \rightarrow \ 0,
\end{array}$$

respectively. Hence we cannot say if Hilbert functions $\mathbf{G_1}$ and $\mathbf{G_2}$ are level only based on shifts and Betti numbers from the above minimal free resolutions of R/I and R/J. In other words, with additional conditions for i to be the maximum value $i = d + 2$ (or $h_d = 2d + 2$ and $h_{d-1} - h_d \geq 5$), we can decide if the Hilbert function in Question 2.1 is ***not*** level using shifts and Betti numbers.

3. ALGORITHMS FOR COCOA TO CONSTRUCT SOME NON-LEVEL O-SEQUENCES.

The folloing algorithm is to check if the given sequence is an O-sequence.

Algorithm 3.1 (CoCoA: Checking O-sequences).

```
Define OSEQUENCE(T)
A1:="==> Yes, this is an O-Sequence.";
A2:="==> No, this is NOT an O-Sequence." ;
A:=A1;
For I:= 2 To Len(T)-1 Do
    J:=I+1;
   S1:=Comp(T,I);
   S2:=Comp(T,J);
        BinValue:=BinExp(S1,I-1,1,1);
--Print I, J, S2, BinValue, NewLine;
If BinValue < S2 Then A:= A2
EndIf;
EndFor;
S:=Comp(T,1);
If S > 1 Then Print "The 1st component should be 1,
         so this is NOT an O-Sequence.", NewLine
```

```
EndIf;
If S < 1 Then Print "The 1st component should be 1,
         so this is NOT an O-Sequence.", NewLine
EndIf;
If S=1 Then  Print T, A
EndIf;
EndDefine;
```

Example 3.2 (CoCoA).

```
(1) OSEQUENCE([1, 6, 19, 26, 41, 18, 17, 14, 13]);
[1, 6, 19, 26, 41, 18, 17, 14, 13]
==> Yes, this is an O-Sequence.

(2) OSEQUENCE([1, 6, 19, 60, 41, 18, 17, 14, 13]);
[1, 6, 19, 60, 41, 18, 17, 14, 13]
==> No, this is NOT an O-Sequence.
```

Note that the truncation of a Gorenstein O-sequence or more generally of a level O-sequence, is again a level O-sequence. Moreover we can construct a Gorenstein O-sequence using a 0-dimensional differentiable O-sequence. Hence we see that any 0-dimensional differentiable O-sequence is always level. The folloing algorithm is to obtain a 0-dimensional differentiable O-sequence which is also level, using the given O-sequence from CoCoA.

Algorithm 3.3 (CoCoA: Adding Up O-sequences)**.**

```
Define ADDUPHilbert(L)
S:=[];
  For I := 2 To Len(L)
    Do S1:=Sum(First(L,I));
    Append(S,S1);
  EndFor;
 S2:=Comp(S,Len(S));
 S:=[S];
 Append(S,S2);
 S;
EndDefine;
```

The following theorem is to obtain a level O-sequence using a given level O-sequence.

Theorem 3.4 (**Theorem 4.8A,** [7])**.** *Let* $h = (h_0, h_1, \ldots, h_e)$ *be the h-vector of a level algebra* $A = R/\mathrm{Ann}(M)$*. Then, if* F *is a generic form of degree* e*, the level algebra* $A = R/\mathrm{Ann}(\langle M, F\rangle)$ *has h-vector* $\mathbf{H} = (H_0, H_1, \ldots, H_e)$*,where, for* $i = 1, \ldots, e$*,*

$$H_i = \min\left\{h_i + \binom{r-1+e-i}{e-i}, \binom{r-1+i}{i}\right\}.$$

We introduce the following algorithm to obtain a level O-sequence using the given level O-sequence based on Theorem 3.4.

Algorithm 3.5 (CoCoA: Obtaining Level h-vector).

```
Define LEVELHVECTOR(T)
NewT:=[1];
R:=Comp(T,2);
E:=Len(T)-1;
For J := 2 To Len(T) Do
 I:=J-1;
        Ti:=Comp(T,J);
 T1:=Bin(R-1+E-I,E-I);
 T2:=Bin(R-1+I,I);
   NewTi:=Min(Ti+T1,T2);
  Append(NewT,NewTi);
EndFor;
Print "From h=", T, "   and r=", R, NewLine;
Print "We have T=", NewT
EndDefine;
```

By the same argument as the proof of Corollary 2.9 in [9], we obtain the following corollary easily.

Corollary 3.6. *Let* $\mathbf{H} = \{h_i\}_{i\geq 0}$ *be an O-sequence with* $h_1 = 6$*. If*

$$h_{d-1} > h_d + 4, \quad h_d \leq 2d + 2, \qquad \text{and} \qquad h_{d+1} \geq h_d$$

for some degree d*, then* $\mathbf{H}$ *is not level.*

Proof. Note that, by the proof of Theorem 2.6, any graded ring with Hilbert function

$$\mathbf{H}' \; : \; h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d \quad \rightarrow$$

has a socle element in degree $d - 1$.

Now let $A = \bigoplus_{i\geq 0} A_i$ be a graded ring with Hilbert function $\mathbf{H}$. If $A_{d+1} = \langle f_1, f_2, \ldots, f_{h_{d+1}} \rangle$ and $I = (f_{h_d+1}, \ldots, f_{h_{d+1}}) \bigoplus_{j\geq d+2} A_j$, then a graded ring $B = A/I$ has Hilbert function

$$h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d,$$

and hence B has a socle element in degree $d-1$ by Theorem 2.6. Since $A_i = B_i$ for every $i \leq d$, A also has the same socle element in degree $d - 1$ as B, and thus $\mathbf{H}$ is not level as we wished. $\square$

Example 3.7. Let $\mathbf{H}$ be as in Example 2.7, which is not level. Note that $14^{\langle 6 \rangle} = 16$.

Hence, by Corollary 3.6, the following two O-sequences

$$\begin{array}{ccccccccc} 1 & 6 & 20 & 40 & 50 & 19 & 14 & 15 & 0, \text{ and} \\ 1 & 6 & 20 & 40 & 50 & 19 & 14 & 16 & 0 \end{array}$$

are not level, either.

Example 3.8. Consider a level O-sequence $(1, 6, 7, 9, 12, 16, 21)$ of codimension 6. By Theorem 3.4, we obtain the following level O-sequence

$$(1, 6, 21, 56, 33, 22, 22)$$

Then $22 = 2 \times 5 + 12$, which shows there exists a level O-sequence of codimension 6 with the condition on $h_d = 2d + 12$

Note that we know that the O-sequence of codimension 6 of type in Question 2.1 is not level if $h_d \leq 2d + 2$ and $h_{d-1} > h_d + 4$. However we can construct an example of a level O-sequence of codimension 6 of type in Question 2.1, when $h_d \geq 2d + 12$.
Unfortunately, it is still open whether an O-sequence of codimension 6 of type in Question 2.1 is level or not, when $h_{d-1} > h_d + 4$ and $2d + 3 \leq h_d \leq 2d + 11$, in general.

Question 3.9. Can we construct a O-sequence of codimension 6 satisfying $h_{d-1} > h_d + 4$ and $2d + 3 \leq h_d \leq 2d + 11$?

4. Inverse System

We now recall an interesting method for construsting Artinian level algebras. This method is based on the idea of ***Macaulay's Inverse Systems***. We will only give a quick review of the method and refer the reader [3] for more details.

Let $R = k[x_1, \cdots, x_n]$ and $S = k[y_1, \cdots, y_n]$. We can consider S as a graded R-module by: if $F \in S_j$ then $x_i \circ F = (\frac{\partial}{\partial y_i})F$. We extend this action in the obvious way and note that the action *lowers* degree on S and hence S is not a finitely generated R-module.

There is an order reversing function from the ideals of R to the R-submodules of S defined by:

$$\varphi_1 : \{\text{ideals of } R\} \rightarrow \{R\text{-submodules of } S\}$$

where

$$\varphi_1(I) = \{F \in S | G \circ F = 0 \text{ for all } G \in I\}$$

This is a 1-1 correspondence whose inverse (φ_2) is given by $\varphi_2(M) = \operatorname{ann}_R(M) = \{r \in R \mid r \cdot x = 0, \forall x \in M\}$. In fact, we denote $\varphi_1(I)$ by I^{-1}, which is called the ***inverse system*** to I.

It is very easy to construct I^{-1} (and this is at the heart of the proof of the 1-1 correspondence). One first observes that the pairing

$$R_j \times S_j \longrightarrow S_0 \simeq k$$

is a perfect pairing and so S_j can be identified with R_j^* (the dual vector space to R_j). If V is a subspace of R_j we write $V^\perp$ for the annihilator of V in this pairing. Then, if $I \subset R$ is an ideal and I_j its j^{th} graded piece, then Macaulay observed that:

$$(I^{-1})_j = I_J^\perp.$$

It follows immediately that

$$\dim_k(I^{-1})_j = \dim_k R_j - \dim_k I_j = \mathbf{H}(R/I, j).$$

It is a simple consequence of this last observation that I^{-1} is a finitely generated R-submodule of S if and only if R/I is Artinian.

Remark 4.1. There is another way to define Inverse Systems which considers S as an R-module in a different way. In this other method, we consider the ***contraction*** operations, D_{x_i} where, if F is a monomial in S_j then

$$D_{x_i}(F) = \begin{cases} 0, & \text{if } y_i \text{ does not divide F,} \\ F/y_i & \text{if } y_i \text{ divides F.} \end{cases}$$

We extend this action to all of S in the obvious way and recall that when the characteristic of k is 0, this action is equivalent to the one described above. The contraction operation has the advantage that it doesn't end up increasing the sizes of coefficients (see [6] for more details).

The really interesting connection between inverse systems and what we've been considering is the following theorem of Macaulay. We continue with notations as above.

Theorem 4.2 (Macaulay)**.** *Let I be an Artinian ideal of R and I^{-1} its inverse system. Then I^{-1} has exactly ν_j minimal generators of degree j if and only if the socle of R/I in degree j has dimension exactly ν_j.*

Remark 4.3. 1) This gives us a new interpretation of the socle vector of an Artinian algebra of the form $A = k[x_1, \cdots, x_n]/I$. The entries of the socle vector tell us the number of generators of the inverse system of I in each degree.

2) Since we are interested in level algebras (Atrinian, say, with socle degree s, type c and embedding dimension n) then this theorem tells us how to make ***all*** of them. We look at every subspace of $S_s = k[y_1, \cdots, y_n]_s$ of dimension c and form the R-submodule of S generated by that subspace. The result is a level algebra of the type we are looking for and every level algebra of socle degree s, type c and embedding dimension n arises in this way.

Example 4.4. Suppose we would like to construct a level algebra with socle degree 5, co-dimension 6 and type 2. Macaulay's Theorem says we have to look at a two dimensional vector space of S_5, where $S = k[y_1, y_2, y_3, y_4, y_5, y_6]$ and take the inverse system it generates.

For example, consider the vector space of S_5 generated by $F_1 = y_1^5$, $F_2 = y_2^5 + y_3^5 + y_4^5 + y_5^5 + y_6^5$ and $F_3 = y_1y_2y_3y_4y_5$.

The inverse system, call it M generated by these three elements of degree 5 will have

$$\begin{aligned}
M_4 &= \langle y_1^4, y_2^4, y_3^4, y_4^4, y_5^4, y_6^4, y_2y_3y_4y_5, y_1y_3y_4y_5, y_1y_2y_4y_5, y_1y_2y_3y_5,\\
&\quad y_1y_2y_3y_4\rangle,\\
M_3 &= \langle y_1^3, y_2^3, y_3^3, y_4^3, y_5^3, y_6^3, y_3y_4y_5, y_2y_4y_5, y_2y_3y_5, y_2y_3y_4, y_1y_4y_5, y_1y_3y_5,\\
&\quad y_1y_3y_4, y_1y_2y_5, y_1y_2y_4, y_1y_2y_3\rangle,\\
M_2 &= \langle y_1^2, y_2^2, y_3^2, y_4^2, y_5^2, y_6^2, y_4y_5, y_3y_5, y_3y_4, y_2y_5, y_2y_4, y_2y_3, y_1y_5, y_1y_4,\\
&\quad y_1y_3, y_1y_2\rangle,\\
M_1 &= \langle y_1, y_2, y_3, y_4, y_5, y_6\rangle, \quad \text{and}\\
M_0 &= \langle 1\rangle.
\end{aligned}$$

So, if $I = \operatorname{ann}_R(M)$ and $A = k[x_1, x_2, x_3, x_4, x_5, x_6]/I$ then the level h-vector of A is $(1, 6, 11, 16, 16, 11, 3)$.

Theorem 4.5 (Theorem 4.4, [1]). *Let $R = k[x_1, x_2, x_3]$ and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the h-vector of a graded Artinian algebra $A = R/I$ with socle degree s. If $h_{d-1} > h_d$ and $h_d = h_{d+1} \le 2d + 3$, then $\mathbf{H}$ is* ***not*** *level.*

Let

$$\mathbf{H} \;:\; h_o \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d \quad \cdots \tag{5}$$

with $h_{d-1} > h_d$.

Theorem 4.6 (Proposition 4.9, Remark 4.10, [1]). *Let*

$$h = (1, 3, 6, \ldots, \overset{(d-1)-\text{st}}{2d + (k-5)}, \overset{d-\text{th}}{2d + (k-3)}, \overset{(d+1)-\text{st}}{2d + (k-1)}).$$

Then we can construct a level O-sequence of codimension 3 *of type in equation* (5) *satisfying*

$$2d + (k+1) = \mathbf{H}_{d-1} > \mathbf{H}_d = \mathbf{H}_{d+1} = 2d + k, \left(5 \le k \le \frac{d^2 - 3d + 2}{2}\right).$$

Remark 4.7. The existence of level h-vectors of codimension 3 immediately implies the existence of level h-vectors of codimension r, for all $r \ge 4$. In fact, if the level algebra $k[x_1, x_2, x_3]/\mathrm{Ann}(M)$ has the h-vector $h = (1, 3, h_2, \ldots, h_e)$ of type (5), then the level algebra

$$k[x_1, \ldots, x_r]/\mathrm{Ann}(\langle M, {y_4}^e, \ldots, {y_r}^e \rangle)$$

has the h-vector of codimension r

$$h' = (1, r, h_2 + r - 3, \ldots, h_e + r - 3),$$

which is clearly of type (5).

Example 4.8 (Example 4.8, [1])**.** Consider a level O-sequence $(1, 3, 5, 7, 9, 11, 13)$ of codimension 3. By Theorem 4.7, we obtain the following level O-sequence:

$$(1, 3, 6, 10, 15, 14, 14)$$

Then $14 = 2 \times 5 + 4$, which shows there exists a level O-sequence of codimension 3 of type in equation (5) when $h_d = 2d + 4$.

Example 4.9. Using Remark 3.7 and Example 3.8, we can obtain a level h-vector of codimension 6 as follows:

$$1 \quad 6 \quad 9 \quad 13 \quad 18 \quad 17 \quad 17.$$

Note that $h_5 = 17 = 2 \times 5 + 7$.

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Abstract

Non Level O-Sequences of codimension 6

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We studied if codimension is 6, whether the level of Artinian O-sequence which is possible to be level is. This study was based on the existing proved codimension which is 4.

At first, the purpose of this study was to show that if $h_{d-1} > h_d + 4$, $h_d = h_{d+1}$ for some $d < s$, $h_d \leq 2d + 2$ and H has a codimension 6, then H is not level.

We also proved that some Non-unimodal Artinian O-sequences of codimension 6 cannot be level.

Moreover, we introduced Algorithm of a computer program CoCoA. We made some examples in this thesis with algorithm.

And we introduced Inverse System for constructing some examples which can be level.