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석사학위청구논문

NON-LEVEL O-SEQUENCES OF CODIMENSION 4

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성신여자대학교 교육대학원

교육학과 수학교육전공

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이 논문을 석사학위논문으로 제출함

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인 준 서

김수현의 석사학위 논문으로 인준함.

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Abstract

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기존에 증명된 여차원이 3 ! 경우에 근거하여 여차원이 4 ! 때, level Artinian O- level 여부를 연구하였다.

우선 $d<s$, $h_d \le 2d+2$ 일 때, $h_{d-1} > h_d + 2$, $h_d = h_{d+1}$ 이고 H 가 여차원이 4 H level .

그리고 여차원이 4 Non-Unimodal Artinian O- level .

또한 Computer Program CoCoA Algorithm 소개하고, Algorithm 을 이용하여 몇 가지 예들을 만들어 보았다.

1. Introduction

Let $R = k[x_0, \ldots, x_n]$ be an n-variable polynomial ring over an infinite field with characteristic 0 and I be a homogeneous ideal of R. Then we have $R = \bigoplus_{i \geq 0} R_i$, where R_i is the vector space of dimension $\binom{i+n}{n}$ generated by all the monomials in R having degree i. Since we can write $I := \bigoplus_{i \geq 0} I_i$, we get

$$A = R/I := \bigoplus_{i \geq 0} (R_i/I_i) = \bigoplus_{i \geq 0} A_i$$

is a graded ring. The ***Hilbert function*** of A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

In this thesis, we shall study Artinian quotients $A = R/I$ of R where I is a homogeneous ideal of R. The ***h-vector*** of A is

$$\mathbf{H} := (h_0, h_1, \ldots, h_s),$$

where $h_i = \dim_k A_i$ and s is the last index such that $\dim_k A_s \neq 0$ and is called the ***socle degree*** of A. The ***socle*** of A is defined by the annihilator of the maximal homogeneous ideal, namely

$$\operatorname{ann}_A(m) := \{a \in A \mid am = 0\} \qquad \text{where} \qquad m = \sum_{i=1}^{s} A_i.$$

Let h and i be positive integers. Then h can be written uniquely in the form

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expansion for h is called the ***i-binomial expansion*** of h. Also, define

$$h^{\langle i\rangle} = \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{(i-1)+1} + \cdots + \binom{m_j+1}{j+1},$$

and $0^{\langle i\rangle} = 0$. Let $\mathbf{T} = (h_0, h_1, \dots, h_i, \dots)$ be a sequence of non-negative integers. We say that $\mathbf{T}$ is an ***O-Sequence*** if $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i\rangle}$ for all $i \geq 1$. Given an O-sequence $\mathbf{T} = (h_0, h_1, \dots, h_i, \dots)$, we can differentiate it to get a new sequence

$$\Delta\mathbf{T} = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \dots)$$

and we call $\Delta\mathbf{T}$ the ***first difference*** of $\mathbf{T}$. If $\Delta\mathbf{T}$ is an O-sequence again, $\mathbf{T}$ is called a ***differentiable O-Sequence***

Let $A = R/I$ be a Cohen-Macaulay ring of dimension d and let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to R \to R/I \to 0$$

be a minimal free resolution of R/I. Then A is a ***level algebra of m*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. In particular, if $m = 1$, then we say that A is a ***Gorenstein algebra*** and the h-vector of A is a ***Gorenstein sequence***. It is well-known that an Artinian graded algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_\ell$ is level if $\operatorname{soc}(A) = A_\ell$.

A total order on the monomials of each degree is said to be a ***term order*** if $x_1 > \cdots > x_n$, and $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m, m_1 and m_2.

The ***reverse lexicographic order*** is a term order defined to be $x_1^{i_1}\cdots x_n^{i_n} > x_1^{j_1}\cdots x_n^{j_n}$ if and only if $\sum i_t > \sum j_t$ or $\sum i_t = \sum j_t$ and there is s such that $i_t = j_t$ for $s < t \leq n$ and $i_s < j_s$.

The ***lexicographic order*** is a term order defined to be $x_1^{i_1}\cdots x_n^{i_n} > x_1^{j_1}\cdots x_n^{j_n}$ if and only if $\sum i_t > \sum j_t$ or $\sum i_t = \sum j_t$ and there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.

Let S be a subset of all monomials in R_d. S is a ***lex-segment*** if a monomial m of deree d is in S, then every monomial m' of degree d in R_d such that $m' > m$ is in S.

Let $I = \oplus_{t\geq 0} I_t$ be a graded ideal of R. We say that I is a ***lex−segment ideal*** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

A monomial ideal I in R is ***stable*** if the monomial

$$\frac{x_j w}{x_{m(w)}}$$

belongs to I for every monomial $w \in I$ and $j < m(w)$ where

$$m(u) := \max\{j \mid a_j > 0\}$$

for $u = x_1^{a_1}\cdots x_n^{a_n}$.

Let S be a subset of all monomials in $R = \bigoplus_{i\geq 0} R_i$ of degree i. We call S a ***Borel fixed set*** if

$$u = x_1^{a_1}\cdots x_n^{a_n} \in S,\ a_i > 0, \quad \text{implies} \quad \frac{x_i u}{x_j} \in S$$

for every $1 \leq i \leq j \leq n$.

A monomial ideal I of R is called a ***Borel fixed ideal*** or ***strongly stable ideal*** if the set of all monomials in I_i is a Borel set for every i. There are two Borel fixed monomial ideals canonically attached to a homogeneous ideal I of R: the generic initial ideal $\mathrm{Gin}(I)$ with respect to the reverse lex order and the lex-segment ideal I^{lex}. The ideal I^{lex} is defined as follows. For the vector space I_d of forms of degree d in I, one defines $(I^{\mathrm{lex}})_d$ to be the vector space generated by largest, in the lexicographical order, $\dim_k(I_d)$ monomials of degree d. By construction, I^{lex} is a strongly stable ideal and it only depends on the Hilbert function of I.

In [7], it has been proved that **H** is not level if $h_{d-1} > h_d = h_{d+1}$ for some $d < s$, $h_d \leq 2d + 2$, and **H** is codimension 3. In this thesis, we show that **H** is not level, when **H** has the same condition and **H** is of codimension 4 (see Theorem 6). We wlso prove that a certain non-unimodal O-sequence of codimension 4 can not be level using Theorem 6

2. Some Non-Level O-sequences of Codimension 4

In [2], they studied the following interesting O-sequence:

$$\mathbf{H} \; : \; h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_{d+1} \quad \cdots \tag{1}$$

where $h_{d-1} > h_d = h_{d+1}$. Hence we have the following interesting question.

Question 1. Let $\mathbf{H}$ be as above. Is $\mathbf{H}$ NOT level?

In this section, we shall prove that a certain O-sequence of codimension 4 is not level.

In [2], it has been proved that $\mathbf{H}$ is not level if $h_d \le d+1$ (see the following proposition).

Proposition 2 (Proposition 2.21, [2])**.** *Let $h = (1, h_1, h_2, \ldots, h_s)$ be the h-vector of an Artinian algebra with socle degree s. Then h is* ***not*** *a level sequence if $h_d = h_{d+1} \le d+1$ and $h_{d-1} > h_d$.*

We shall revise the above proposition to a case in codimension 4 with $h_{d-1} > h_d + 2$ and $h_d \le 2d + 2$.

Let $R = k[x_1, \dots, x_n]$ and $A = R/I$ where I is a homogeneous ideal of R having height n. Then A has the minimal free resolution $\mathcal{F}$, as an R-module, of the form:

$$0 \;\rightarrow\; \mathcal{F}_{n-1} \;\rightarrow\; \cdots \;\rightarrow\; \mathcal{F}_1 \;\rightarrow\; \mathcal{F}_0 \;\rightarrow\; R \;\rightarrow\; A \;\rightarrow\; 0$$

where $\mathcal{F}_j = \bigoplus_{t=1}^{\gamma_j} R^{\beta_{j,j+1+t}}(-(j+1+t))$ are each free graded R-modules.

The number $\{\beta_{j,i}\}$, for fixed $\jmath$, $0 \leq \jmath \leq n-1$, are called the $\jmath^{th}$ ***graded Betti numbers*** of the ideal I. It is well-known that the socle vector of A, $s(A) = (a_1, \dots, a_s)$, is related to the $(n-1)^{st}$ graded Betti numbers as follows:

$$\beta_{n-1,n+i} = a_i.$$

It follows that A is a level algebra if and only if $\beta_{n-1,n+i} = 0$ for all $i \neq s$.

For I as above, the ***Betti diagram*** of R/I is a useful device to encode the graded Betti numbers of R/I (and hence of I). It is constructed as follows:

$$\begin{array}{c|cccccc}
 & & 0 & 1 & \dots & n-1 \\
0 & 1 & 0 & 0 & \dots & 0 \\
1 & 0 & * & * & \dots & * \\
t & 0 & \beta_{0,t+1} & \beta_{1,t+2} & * & \beta_{n-1,t+n} \\
d-2 & 0 & \beta_{0,d-1} & \beta_{1,d} & * & \beta_{n-1,d-2+n} \\
d-1 & 0 & \beta_{0,d} & \beta_{1,d+1} & * & \beta_{n-1,d-1+n} \\
d & 0 & \beta_{0,d+1} & \beta_{1,d+2} & * & \beta_{n-1,d+n}
\end{array}$$

Eliahou and Kervaire [1] studied minimal free resolutions of certain monomial ideals. We recall some of their results now.

Theorem 3 (Eliahou–Kervaire, [1])**.** *Let I be a stable monomial ideal of R (e.g., a lex segment ideal). Denote by $\mathcal{G}(I)_d$ the elements of that set which have degree d. Then*

$$\beta_{q,i} = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T)-1}{q}.$$

This Theorem gives all the graded Betti numbers of the lex segment ideal just from an intimate knowledge of the generators of that ideal. Since the minimal free resolution of the ideal of a k-configuration in $\mathbb{P}^n$ is extremal ([3], [4]), we may apply this result to those ideals. It is an immediate consequence of the Eliahou–Kervaire Theorem that if I is either a lex-segment ideal or the ideal of a k-configuration in $\mathbb{P}^n$ which has *no* generators of degree d, then $\beta_{q,i} = 0$ whenever $i - q = d$.

By the result of [6], the only way we can cancel graded Betti numbers is if there are the same graded Betti numbers in the adjacent free modules of the extremal minimal free resolution. Note that it is quite obvious for a case of $n = 3$.

By the same idea as the proof of Proposition 2.5 in [7], we obtain the following proposition.

Lemma 4. *Let I be the lex-segment ideal in $R = k[x_1, x_2, x_3, x_4]$ with Hilbert function $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ where $h_d = d+i$ and $1 \leq i \leq \frac{d^2+d}{2}$. Then the last monomial of I_d is*

$$\begin{array}{ccrcccl} x_2x_3^{i-1}x_4^{d-i}, & \textit{for} & 1 & \leq & i & \leq & d, \\ x_2^2x_3^{i-(d+1)}x_4^{(2d-1)-i}, & \textit{for} & d+1 & \leq & i & \leq & 2d-1, \\ \vdots & & & & & & \\ x_2^{d-1}x_3^{i-\frac{d^2+d-4}{2}}x_4^{\frac{d^2+d-2}{2}-i}, & \textit{for} & \frac{d^2+d-4}{2} & \leq & i & \leq & \frac{d^2+d-2}{2}, \\ x_2^d, & \textit{for} & & & i & = & \frac{d^2+d}{2}. \end{array}$$

Proposition 5. *Let $R = k[x_1, x_2, x_3, x_4]$ and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the h-vector of an Artinian algebra with socle degree s and*

$$h_d = h_{d+1} = d+i, \quad h_{d-1} > h_d, \quad \textit{and} \quad j := h_{d-1} - h_d$$

for $i = 1, 2, \ldots, \frac{d^2+d}{2}$. Then, for every $1 \leq k \leq d$ and $0 \leq \ell \leq d$,

$$\beta_{2,d+3} = \begin{cases} 3k-2, & \textit{for } (k-1)d - \frac{k(k-3)}{2} \leq i \leq (k-1)d - \frac{k(k-3)}{2} + (k-1), \\ 3k, & \textit{for } (k-1)d - \frac{k(k-3)}{2} + k \leq i \leq kd - \frac{(k-1)k}{2}. \end{cases}$$

$$\beta_{3,d+3} = j + \ell, \qquad \textit{for } (\ell-1)d - \frac{(\ell-2)(\ell-1)}{2} < i \leq \ell d - \frac{(\ell-1)\ell}{2}.$$

Proof. Since we assume $h_d = d + i$, the monomials not in I_d are the last $d+i$ monomials of R_d. By Lemma 4, the last monomial of R_1I_d is

$$\begin{array}{ccl} x_2x_3^{i-1}x_4^{d-i+1}, & \text{for} & i = 1, \ldots, d, \\ x_2^2x_3^{i-(d+1)}x_4^{2d-i}, & \text{for} & i = d+1, \ldots, 2d-1, \\ \vdots & & \\ x_2^{d-1}x_3^{i-\frac{d^2+d-4}{2}}x_4^{\frac{d^2+d}{2}-i}, & \text{for} & i = \frac{d^2+d-4}{2}, \ \frac{d^2+d-2}{2}, \\ x_2^dx_4, & \text{for} & i = \frac{d^2+d}{2}. \end{array}$$

In what follows, the first monomial of $I_{d+1} - R_1 I_d$ is

$$\begin{array}{ll} x_3^{d+1}, & \text{for} \quad i = 1, \\ x_2 x_3^{i-2} x_4^{(d+2)-i}, & \text{for} \quad i = 2, \ldots, d, \\ \vdots & \\ x_2^{d-1} x_3 x_4, & \text{for} \quad i = \frac{d^2+d-2}{2}, \\ x_2^{d-1} x_3^2, & \text{for} \quad i = \frac{d^2+d}{2}. \end{array} \tag{2}$$

Note that

$$\begin{array}{rcl} (d+i)^{\langle d \rangle} & = & (d+i)+k, \quad \text{for } \ i = (k-1)d - \frac{k(k-3)}{2}, \ldots, kd - \frac{k(k-1)}{2}, \\ & & \qquad\qquad\quad \text{and } k = 1, \ldots, d. \end{array} \tag{3}$$

We now calculate the Betti number

$$\beta_{2,d+3} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T)-1}{2}.$$

Based on equation (2), we shall find this Betti number of two cases for i as follows.

Case 1-1. $i = (k-1)d - \frac{k(k-3)}{2}$ and $k = 1, 2, \ldots, d$.

Then, by equation (3), I_{d+1} has k-generators, which are

$$x_2^{k-1} x_3^{(d+2)-k}, x_2^{k-1} x_3^{(d+1)-k} x_4, \ldots, x_2^{k-1} x_3^{(d+3)-2k} x_4^{k-1}.$$

By the similar argument, I_{d+1} has k-generators including the element $x_2^{k-1} x_3^{(d+2)-k}$ for $i = (k-1)d - \frac{k(k-3)}{2} + 1, \ldots, (k-1)d - \frac{k(k-3)}{2} + (k-1)$. Hence we have that

$$\beta_{2,d+3} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T)-1}{2} = 3 \times (k-1) + 1 = \underline{3k-2}.$$

Case 1-2. $i = (k-1)d - \frac{k(k-3)}{2} + k = (k-1)d - \frac{k(k-5)}{2}, \dots, kd - \frac{k(k-1)}{2}$ and $k = 1, 2, \dots, d$.

Then, by equation (3), I_{d+1} has k-generators, which are

$$x_2^k x_3^{i- \ (k-1)d - \frac{k^2-3k-2}{2}} \ x_4^{kd - \frac{k^2-k-4}{2} - i}, \dots, x_2^k x_3^{i - \left((k-1)d - \frac{k(k-5)}{2}\right)} x_4^{\left(kd - \frac{k(k-3)}{2} + 1\right) - i}.$$

Hence we have that

$$\beta_{2,d+3} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T)-1}{2} = 3 \times k = \underline{3k}.$$

Now we move on the Betti number:

$$\beta_{3,d+3} = \sum_{T \in \mathcal{G}(I)_{d}} \binom{m(T)-1}{3}.$$

Recall $h_d = d + i$ and $j := h_{d-1} - h_d$. The calculation in this case is much more complicated, and there are four cases based on i and j.

Case 2-1. $(\ell-1)d - \frac{(\ell-2)(\ell-1)}{2} < i < \ell d - \frac{(\ell-1)\ell}{2}$ and $\ell = 1, 2, \dots, d$.

Then the last monomial of I_d is

$$x_2^\ell x_3^{i-(\ell-1)d + \frac{\ell(\ell-3)}{2}} x_4^{\ell d - \frac{(\ell-1)\ell}{2} - i}.$$

(a) $(k-1)d - \frac{(k-1)k}{2} < i + j < kd - \frac{k(k+1)}{2}$ and $k = \ell, \ell+1, \dots, d$.

Then the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_2^k x_3^{(i+j) - \left((k-1)d - \frac{(k-2)(k+1)}{2}\right)} x_4^{\left(kd - \frac{(k-1)(k+2)}{2}\right) - (i+j)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$x_2^k x_3^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_4^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)}, \ldots, x_2^k x_4^{d-k},$
$x_2^{(k-1)} x_3^{d-(k-1)}, x_2^{(k-1)} x_3^{(d-1)-(k-1)} x_4, \ldots, x_2^{(k-1)} x_4^{d-(k-1)},$
$\vdots$
$x_2^{\ell+1} x_3^{(d-1)-\ell}, x_2^{\ell+1} x_3^{(d-2)-\ell} x_4, \ldots, x_2^{\ell+1} x_4^{(d-1)-\ell}$
$x_2^{\ell} x_3^{d-\ell}, \ldots, x_2^{\ell} x_3^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}} x_4^{\ell d-\frac{(\ell-1)\ell}{2}-i}$

and thus

$$\beta_{3,d+3} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T)-1}{3} = \underline{j+\ell},$$

(b) $i+j = (k-1)d - \frac{(k-1)k}{2}$ and $k = \ell+1, \ldots, d$.

Then the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_2^{k-1} x_3^{d-(k-1)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$x_2^{k-1} x_3^{d-(k-1)}, x_2^{k-1} x_3^{(d-1)-(k-1)} x_4, \ldots, x_2^{k-1} x_4^{d-(k-1)},$
$\vdots$
$x_2^{\ell+1} x_3^{(d-1)-\ell}, x_2^{\ell+1} x_3^{(d-2)-\ell} x_4, \ldots, x_2^{\ell+1} x_4^{(d-1)-\ell}$
$x_2^{\ell} x_3^{d-\ell}, \ldots, x_2^{\ell} x_3^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}} x_4^{\ell d-\frac{(\ell-1)\ell}{2}-i}$

and thus

$$\beta_{3,d+3} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T)-1}{3} = \underline{j+\ell}.$$

Case 2-2. $i = \ell d - \frac{(\ell-1)\ell}{2}$ and $\ell = 1, 2, \ldots, d$.

Then the last monomial of I_d is

$$x_2^{\ell} x_3^{d-\ell}.$$

(a) $(k-1)d-\frac{(k-1)k}{2} < i+j < kd-\frac{k(k+1)}{2}$ and $k=\ell+1,\dots,d$.

Then the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_2^k x_3^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_4^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$\begin{array}{l} x_2^k x_3^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_4^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)}, \dots, x_2^k x_4^{d-k}, \\ x_2^{(k-1)} x_3^{d-(k-1)}, x_2^{(k-1)} x_3^{(d-1)-(k-1)} x_4, \dots, x_2^{(k-1)} x_4^{d-(k-1)}, \\ \qquad\qquad \vdots \\ x_2^{\ell+1} x_3^{(d-1)-\ell}, x_2^{\ell+1} x_3^{(d-2)-\ell} x_4, \dots, x_2^{\ell+1} x_4^{(d-1)-\ell} \\ x_2^{\ell} x_3^{d-\ell}, \end{array}$$

and thus

$$\beta_{3,d+3} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{3} = \underline{j+\ell}.$$

(b) $i+j=(k-1)d-\frac{(k-1)k}{2}$ and $k=\ell+1,\dots,d$.

Then the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_3^{(k-1)} x_4^{d-(k-1)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$\begin{array}{l} x_2^{(k-1)} x_3^{d-(k-1)}, x_2^{(k-1)} x_3^{(d-1)-(k-1)} x_4, \dots, x_2^{(k-1)} x_4^{d-(k-1)}, \\ \qquad\qquad \vdots \\ x_2^{\ell+1} x_3^{(d-1)-\ell}, x_2^{\ell+1} x_3^{(d-2)-\ell} x_4, \dots, x_2^{\ell+1} x_4^{(d-1)-\ell} \\ x_2^{\ell} x_3^{d-\ell}, \end{array}$$

and thus

$$\beta_{3,d+3} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{3} = \underline{j+\ell},$$

as we wished. $\square$

Theorem 6. *Let* **H** *and j be as in Proposition 5. Then for every $i = -(d-1), \ldots, -1, 0, 1, 2, \ldots, d, d+1, d+2$, and $j \geq 3$,* **H** *is not level.*

Proof. By Proposition 3.8 in [2], this proposition holds for $i = -(d-1), \ldots, -1, 0, 1$. Hence we may assume that $2 \leq i \leq d+2$.

Let J be a lex-segment ideal in $R = k[x_1, x_2, x_3, x_4]$ such that the Hilbert function R/J is **H**.

Then, by Proposition 5, we have that

$$\begin{aligned} \beta_{2,d+3}(J) &= \begin{cases} 3, & \text{for} \quad i = 2, \ldots, d, \\ 4, & \text{for} \quad i = d+1, d+2, \end{cases} \qquad \text{and} \\ \beta_{3,d+3}(J) &= \begin{cases} j+1, & \text{for} \quad i = 2, \ldots, d, \\ j+2, & \text{for} \quad i = d+1, d+2. \end{cases} \end{aligned} \tag{4}$$

Hence if $j \geq 3$, then we have $\beta_{3,d+3} > \beta_{2,d+3}$. Therefore **H** is not level, as we wished. □

Example 7. Consider an Artinian O-sequence

$$\mathbf{H} \ : \ 1 \quad 4 \quad 10 \quad 13 \quad 10 \quad 10 \quad 0.$$

Then $h_d = h_4 = 2\times 4+2 = 10$ and $h_{d-1} - h_d = h_3 - h_4 = 13 - 10 = 3$. Hence, by Theorem 6, **H** is not level.

Example 8. Consider an Artinian O-sequences

$$\begin{array}{lcccccccc} \mathbf{G_1} & : & 1 & 4 & 10 & 12 & 10 & 10 & 0. \\ \mathbf{G_2} & : & 1 & 4 & 10 & 14 & 11 & 11 & 0. \end{array}$$

which do not satisfy the condition of Theorem 6.

Let I and J be lex-segment ideals in $R = k[x_1, x_2, x_3, x_4]$ such that the Hilbert functions of R/I and R/J are $\mathbf{G_1}$ and $\mathbf{G_2}$, respectively. Then the minimal free resolutions of R/I and R/J are

$$\begin{array}{cl}
0 & \rightarrow \; R^2(-6) \oplus \mathbf{R^4(-7)} \oplus R(-8) \oplus R^{10}(-9) \\
 & \rightarrow \; R^9(-5) \oplus R^{14}(-6) \oplus \mathbf{R^4(-7)} \oplus R^{31}(-8) \\
 & \rightarrow \; R^{14}(-4) \oplus R^{17}(-5) \oplus R^5(-6) \oplus R^{32}(-7) \\
 & \rightarrow \; R^8(-3) \oplus R^7(-4) \oplus R^2(-5) \oplus R^{11}(-6) \\
 & \rightarrow \; R \; \rightarrow \; R/I \; \rightarrow \; 0, \qquad \text{and}
\end{array}$$

$$\begin{array}{cl}
0 & \rightarrow \; R(-6) \oplus \mathbf{R^5(-7)} \oplus R^2(-8) \oplus R^{11}(-9) \\
 & \rightarrow \; R^5(-5) \oplus R^{18}(-6) \oplus \mathbf{R^6(-7)} \oplus R^{35}(-8) \\
 & \rightarrow \; R^9(-4) \oplus R^{22}(-5) \oplus R^6(-6) \oplus R^{37}(-7) \\
 & \rightarrow \; R^6(-3) \oplus R^9(-4) \oplus R^2(-5) \oplus R^{13}(-6) \\
 & \rightarrow \; R \; \rightarrow \; R/J \; \rightarrow \; 0,
\end{array}$$

respectively. Hence we cannot say if Hilbert functions $\mathbf{G_1}$ and $\mathbf{G_2}$ are level only based on shifts and Betti numbers from the above minimal free resolutions of R/I and R/J. In other words, with an additional condition for i to be the maximum value $i = d + 2$ (or $h_d = 2d + 2$), we can decide if the Hilbert function in Question 1 is ***not*** level using shifts and Betti numbers.

3. Algorithms for CoCoA to Construct Some Non-Level O-sequences.

The folloing algorithm is to check if the given sequence is an O-sequence.

Algorithm 9 (CoCoA: Checking O-sequences).

```
Define OSEQUENCE(T)
A1:="==> Yes, this is an O-Sequence.";
A2:="==> No, this is NOT an O-Sequence." ;
A:=A1;
For I:= 2 To Len(T)-1 Do
    J:=I+1;
   S1:=Comp(T,I);
   S2:=Comp(T,J);
        BinValue:=BinExp(S1,I-1,1,1);
--Print I, J, S2, BinValue, NewLine;
If BinValue < S2 Then A:= A2
EndIf;
EndFor;
S:=Comp(T,1);
If S > 1 Then Print "The 1st component should be 1,
         so this is NOT an O-Sequence.", NewLine
EndIf;
```

```
  If S < 1 Then Print "The 1st component should be 1,
           so this is NOT an O-Sequence.", NewLine
  EndIf;
  If S=1 Then  Print T, A
  EndIf;
  EndDefine;
```

Example 10 (CoCoA).

```
(1) OSEQUENCE([1, 4, 9, 13, 17, 9, 4]);
[1, 4, 9, 13, 17, 9, 4]==> Yes, this is an O-Sequence.

(2) OSEQUENCE([1, 4, 9, 13, 20]);
[1, 4, 9, 13, 20]==> No, this is NOT an O-Sequence.
```

Note that the truncation of a Gorenstein O-sequence or more generally of a level O-sequence, is again a level O-sequence. Moreover we can construct a Gorenstein O-sequence using a 0-dimensional differentiable O-sequence. Hence we see that any 0-dimensional differentiable O-sequence is always level. The folloing algorithm is to obtain a 0-dimensional differentiable O-sequence which is also level, using the given O-sequence from CoCoA.

Algorithm 11 (CoCoA: Adding Up O-sequences)**.**

```
Define ADDUPHilbert(L)
S:=[];
  For I := 2 To Len(L)
    Do S1:=Sum(First(L,I));
    Append(S,S1);
  EndFor;
 S2:=Comp(S,Len(S));
 S:=[S];
 Append(S,S2);
 S;
EndDefine;
```

The following Theorem is to obtain a level O-sequence using the given level O-sequence.

Theorem 12 (**Theorem 4.8A,** [5])**.** *Let* $h = (h_0, h_1, \ldots, h_e)$ *be the h-vector of a level algebra* $A = R/\mathrm{Ann}(M)$*. Then, if* F *is a generic form of degree* e*, the level algebra* $A = R/\mathrm{Ann}(\langle M, F\rangle)$ *has h-vector* $\mathbf{H} = (H_0, H_1, \ldots, H_e)$*,where, for* $i = 1, \ldots, e$*,*

$$H_i = \min\left\{ h_i + \binom{r-1+e-i}{e-i}, \binom{r-1+i}{i} \right\}.$$

We introduce the following algorithm to obtain a level O-sequence using the given level O-sequence based on Theorem 12.

Algorithm 13 (CoCoA: Obtaining Level h-vector)**.**

```
Define LEVELHVECTOR(T)
NewT:=[1];
R:=Comp(T,2);
E:=Len(T)-1;
For J := 2 To Len(T) Do
 I:=J-1;
        Ti:=Comp(T,J);
 T1:=Bin(R-1+E-I,E-I);
 T2:=Bin(R-1+I,I);
   NewTi:=Min(Ti+T1,T2);
  Append(NewT,NewTi);
EndFor;
Print "From h=", T, "   and r=", R, NewLine;
Print "We have T=", NewT
EndDefine;
```

By the same argument as the proof of Corollary 2.9 in [7], we obtain the following corollary easily.

Corollary 14. *Let* $\mathbf{H} = \{h_i\}_{i\geq 0}$ *be an O-sequence with* $h_1 = 4$*. If*

$$h_{d-1} > h_d + 2, \quad h_d \leq 2d+2, \qquad \text{and} \qquad h_{d+1} \geq h_d$$

for some degree d*, then* $\mathbf{H}$ *is not level.*

Proof. Note that, by the proof of Theorem 6, any graded ring with Hilbert function

$$\mathbf{H}' \ : \ h_0 \ \ h_1 \ \ \cdots \ \ h_{d-1} \ \ h_d \ \ h_d \ \rightarrow$$

has a socle element in degree $d-1$.

Now let $A = \bigoplus_{i\geq 0} A_i$ be a graded ring with Hilbert function $\mathbf{H}$. If $A_{d+1} = \langle f_1, f_2, \ldots, f_{h_{d+1}} \rangle$ and $I = (f_{h_d+1}, \ldots, f_{h_{d+1}}) \bigoplus_{j\geq d+2} A_j$, then a graded ring $B = A/I$ has Hilbert function

$$h_0 \ \ h_1 \ \ \cdots \ \ h_{d-1} \ \ h_d \ \ h_d,$$

and hence B has a socle element in degree $d-1$ by Theorem 6. Since $A_i = B_i$ for every $i \leq d$, A also has the same socle element in degree $d-1$ as B, and thus $\mathbf{H}$ is not level as we wished. $\square$

Example 15. Let $\mathbf{H}$ be as in Example 7, which is not level. Note that $10^{\langle 4\rangle} = 12$. Hence, by Corollary 14, the following two O-sequences

$$\begin{array}{lllllll} 1 & 4 & 10 & 13 & 10 & 11 & 0, \ \text{and} \\ 1 & 4 & 10 & 13 & 10 & 12 & 0. \end{array}$$

are not level, either.

Example 16. Consider a level O-sequence $(1, 4, 7, 10, 11, 13, 16)$ of codimension 4. By Theorem 12, we obtain the following level O-sequence

$$(1, 4, 10, 20, 21, 17, 17)$$

Then $17 = 2 \times 5 + 7$, which shows there exists a level O-sequence of codimension 4 with the condition on $h_d = 2d + 7$

Note that we know that the O-sequence of codimension 4 of type in Question 1 is not level if $h_d \leq 2d + 2$ and $h_{d-1} > h_d + 2$. However we can construct an example of a level O-sequence of codimension 4 of type in Question 1, when $h_d \geq 2d + 7$. Unfortunately, we don't know if an O-sequence of codimension 4 of type in Question 1 is level, when $h_{d-1} > h_d + 2$ and $2d + 3 \leq h_d \leq 2d + 6$.

Question 17. Can we construct a O-sequence of codimension 4 when $h_{d-1} > h_d + 2$ and $2d + 3 \leq h_d \leq 2d + 6$?

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ABSTRACT

ARTINIAN O-SEQUENCE OF SOCLE DEGREE 6 AND TYPE 3

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Based on a theorem of Fröberg and Laksov in [7], 334 Codimension 3, Type 3, and Length 7 Artinian O-sequences was discovered by CoCoA (see Appendix A). Of them, after that, the case, that is not become a level, was studied .

After the O-sequence, is not become a level by Theorem and Remark, was found out, the table, which conforms to each case, was completed. Of the O-sequence that was not rejected by the former method, the special case, that is not become a level, was also demonstrated.

As a result, it was demonstrated that 221 O-sequences, of 334 O-sequences, is not a level. 10 O-sequences, that can not know whether they are a level or not by the former method, was remained.