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석사학위 청구논문

Non-Level Artinian O-sequences of codimension 3 and type 4

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성신여자대학교 교육대학원

교육학과 수학교육전공

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논 문 개 요

Fröberg and Laksov 의 정리에 근거하여 여차원이 3, 형태가 4, 길이가 7인 level이 될 가능성이 있는 291개의 Artinian O-수열에 대하여 level 여부를 연구하였다.

더불어, 참고논문 7번에서 D.S.Shin의 알고리즘 4와 유사한 알고리즘과 “link-sum” 구성을 사용하여 참고논문 1번에서 E.Y.Cho와 16번에서 I.S.Yoo에 의해 주어진 위의 291개의 경우 중 몇 가지를 구성하는 방법에 대해 소개한다.

하지만 여전히 level 여부를 알 수 없는 O-수열 30개가 남았다.

CONTENTS

ABSTRACT

Non-Level Artinian O-sequences of codimension 3 and type 4

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In this thesis, we try to verify that 291 codimension 3 Artinian O-sequences of type 4 and length 7 are level or not.

Moreover, using the similar algorithm to Algorithm 4 in [7] and ''link-sum" construction, we introduce

how to construct some of the above 291 cases given by [1] and [16].

Unfortunately, there are still 30 unknown cases which we should know.

1. **Introduction**

Let $R = k[x_1, \ldots, x_n] = \oplus_{i \geq 0} R_i$, k an algebraically closed field of characteristic 0, and let I be a homogeneous ideal of R, $A = R/I$. The ***Hilbert function*** of A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

In case I is the ideal of a subscheme, $\mathbb{X}$ of $\mathbb{P}^n$, the Hilbert function of $A = R/I$ is sometimes denoted $\mathbf{H}\ (-)$.

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R. The ***h-vector*** of A is $h(A) = (h_0, h_1, \ldots, h_\ell)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and ℓ is the last index such that $\dim_k A_k \neq 0$. We call ℓ the ***socle degree*** A. Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the ***codimension*** of A.

Now we recall a special case of a theorem of Fröberg and Laksov of [9] that if A is a level Artinian algebra of codimension 3 and type m with $\sigma - 1 = t$, then

$$\mathbf{H}(A, i) \leq \min \left\{ \binom{2+i}{2}, m\binom{t-i+2}{2} \right\}.$$

Hence, we have all possible 291 codimension 3 level Artinian O-sequences of type 4 and length 7 using Algorithm in [8]. Thanks to Georgio Dalzzoto for the algorithm.

In this thesis, we try to verify whose O-sequences are level or not. In particular, we prove that some Artinian algebra $A = R/I$ cannot be level when either $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) + 1$ or $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$ (see Propositions 3.3 and 3.6). Morever, using the similar algorithm to Algorithm 4 in [7] and "link-sum" construction, we introduce how to construct some of the above 291 cases given by [1] and [16] (see Appendix A). Unfortunately, there are still 30 unknown cases which we should know (see Appendix B).

2. Non-Level Artinian O-sequences of codimension 3 and type 4

First of all, we introduce some definitions and preliminary results here.

Definition 2.1. (a) Let $h, i > 0$.

Then $h = \binom{m_i}{i} + \binom{m_i - 1}{i-1} + \cdots + \binom{m_j}{j}$ with $m_i > m_i - 1 > \cdots > m_j \geq 1$

is called ***i-th binomial expansion***.

(b) The sequnce $\{h_i\}_{i\geq 0}$ $(h_i \geq 0)$ is called an ***O-Sequence*** if

$h_{i+1} \leq h_i^{\langle i\rangle},\ h_0 = 1,\ i \geq 1.$

Definition 2.2. Let $A = \bigoplus_{i\geq 0} A_i$ be graded ring. Then $A = k[A_1]$ is called ***G-algebra*** if A_1 is a finitely generated vector space over k.

Theorem 2.3 (Macaulay)**.** (a) *$\{h_i\}_{i\geq 0}$ is an O-Sequence.*

(b) *$\{h_i\}_{i\geq 0}$ is the Hilbert function for some standard G-algebra.*

Definition-Proposition 2.4 (Definition-Proposition 2.21, [5])**.** Let $R = k[x_0, \ldots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. Then

(a) If $B = B_0 \oplus \cdots \oplus B_\ell$ $(B_\ell \neq 0)$ is an Artinian algebra, then B is ***level*** if and only if $B_\ell = \text{Ann}(B_1)$.

(b) A is a ***level algebra*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. rank $\mathcal{F}_{n-(d-1)}$
$=$ Cohen-Macaulay type of A.

(c) i) If $\mathbb{X}$ is a non-degenerate set of points in $\mathbb{P}^n$, $A = R/I$ its coordinate ring, then we say that ℓ is the ***socle degree*** of $\mathbb{X}$ if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where $\overline{L}$ is any linear non-zero-divisor of A.

ii) $\mathbb{X}$ is called a ***level set*** of points if $A = R/I$ is a level algebra. In this case, the socle degree of $\mathbb{X}$ is $\ell = \sigma(\mathbb{X}) + n - 1$.

(d) If $\overline{L}$ is a linear non-zero divisor in $A = R/I$, then A is level if and only if $A/\overline{L}A \simeq A/(L, I\ \ )$ is level.

(e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an

Artinian algebra) $b = \{b_i\}_{i \geq 0}$ with $b_1 = n + 1$ is called ***level*** if there is a level set of points in with Hilbert function b.

Definition 2.5. (a) A total order on the monomials of each degree is said to be a ***term order*** if

i) $x_1 > \cdots > x_n$, and

ii) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m, m_1 and m_2.

(b) The ***reverse lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $s < t \leq n$ and $i_s < j_s$.

(c) The ***lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.

(d) Let S be a subset of all monomials in R_d. S is a ***lex-segment*** if a monomial m of deree d is in S, then every monomial m' of degree d in R_d such that $m' > m$ is in S.

(e) Let $I = \bigoplus_{t \geq 0} I_t$ be a graded ideal of R. We say that I is a ***lex-segment ideal*** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

Theorem 2.6 (The Cancelation Principle, [10], [12]). *For any homogeneous ideal I and any i and d, there a complex of $k \cong R/m-$modules $V_\bullet^d$ such that*

$$\begin{array}{rcl} V_i^d & \cong & \operatorname{Tor}_i^R(\operatorname{in}(I), k)_d \\ H_i(V_\bullet^d) & \cong & \operatorname{Tor}_i^R(I, k)_d. \end{array}$$

Remark 2.7. One way to paraphrase this theorem is to say that the minimal free resolution of I is obtained from that $\operatorname{in}(I)$, the *initial ideal* of I, by canceling some adjacent terms of the same degree.

Example 2.8. Let I be a lex-segment ideal of $R = k[x, y, z]$ with the Hilbert function 1 3 5 6 4 4 4. Then the minimal free resolution of R/I is

$$\begin{array}{rl} 0 & \to R^2(-6) \oplus R^4(-9) \\ & \to R(-4) \oplus R^5(-5) \oplus R^8(-8) \\ & \to R(-2) \oplus R(-3) \oplus R^3(-4) \oplus R^4(-7) \\ & \to R \to R/I \to 0. \end{array}$$

Hence the two copies $R(-6)$ of the last free module cannot be cancelled, and thus the above O-sequence cannot be level.

Using the same idea as above, we can show that the following 82 cases in Table 1 are not level.

17)	1 3 5 4 4 4 4	21)	1 3 5 5 4 4 4	29)	1 3 5 6 4 4 4
31)	1 3 5 6 5 5 4	42)	1 3 5 7 4 4 4	44)	1 3 5 7 5 5 4
56)	1 3 5 7 8 5 4	62)	1 3 5 7 9 5 4	69)	1 3 6 4 4 4 4
73)	1 3 6 5 4 4 4	81)	1 3 6 6 4 4 4	83)	1 3 6 6 5 5 4
90)	1 3 6 6 7 5 4	92)	1 3 6 6 7 7 4	93)	1 3 6 6 7 8 4
94)	1 3 6 7 4 4 4	96)	1 3 6 7 5 5 4	108)	1 3 6 7 8 5 4
121)	1 3 6 8 4 4 4	123)	1 3 6 8 5 5 4	127)	1 3 6 8 6 6 4
135)	1 3 6 8 8 5 4	139)	1 3 6 8 8 9 4	141)	1 3 6 8 9 5 4
149)	1 3 6 8 10 5 4	157)	1 3 6 9 4 4 4	159)	1 3 6 9 5 5 4
162)	1 3 6 9 6 5 4	163)	1 3 6 9 6 6 4	168)	1 3 6 9 7 7 4
171)	1 3 6 9 8 5 4	175)	1 3 6 9 8 9 4	177)	1 3 6 9 9 5 4
182)	1 3 6 9 9 10 4	185)	1 3 6 9 10 5 4	194)	1 3 6 9 11 5 4
195)	1 3 6 9 11 6 4	203)	1 3 6 9 12 5 4	204)	1 3 6 9 12 6 4
211)	1 3 6 10 4 4 4	213)	1 3 6 10 5 5 4	216)	1 3 6 10 6 5 4
217)	1 3 6 10 6 6 4	222)	1 3 6 10 7 7 4	225)	1 3 6 10 8 5 4
228)	1 3 6 10 8 8 4	229)	1 3 6 10 8 9 4	231)	1 3 6 10 9 5 4
236)	1 3 6 10 9 10 4	239)	1 3 6 10 10 5 4	248)	1 3 6 10 11 5 4
249)	1 3 6 10 11 6 4	257)	1 3 6 10 12 5 4	258)	1 3 6 10 12 6 4
266)	1 3 6 10 13 5 4	267)	1 3 6 10 13 6 4	275)	1 3 6 10 14 5 4
276)	1 3 6 10 14 6 4	277)	1 3 6 10 14 7 4	284)	1 3 6 10 15 5 4
285)	1 3 6 10 15 6 4	286)	1 3 6 10 15 7 4		

TABLE 1. Non-cancelable 82-cases.

Theorem 2.9 (Theorem 4.1, [2]). *Let* $R = k[x_1, x_2, x_3]$ *and let* $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ *be the h-vector of a graded Artinian algebra* $A = R/I$ *with socle degree s. If*

$$h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \leq 2d + 3$$

then **H** *is* **not** *level.*

Using Theorem 2.9, the following 82 cases in Table 2 cannot be level.

2)	1 3 3 4 5 4 4	3)	1 3 4 5 5 4 4	6)	1 3 4 4 5 4 4
10)	1 3 4 5 5 4 4	13)	1 3 4 5 6 4 4	18)	1 3 5 4 5 4 4
21)	1 3 5 5 4 4 4	22)	1 3 5 5 5 4 4	25)	1 3 5 5 6 4 4
27)	1 3 5 5 6 6 4	29)	1 3 5 6 4 4 4	30)	1 3 5 6 5 4 4
31)	1 3 5 6 5 5 4	33)	1 3 5 6 6 4 4	37)	1 3 5 6 7 4 4
42)	1 3 5 7 4 4 4	43)	1 3 5 7 5 4 4	44)	1 3 5 7 5 5 4
46)	1 3 5 7 6 4 4	48)	1 3 5 7 6 6 4	50)	1 3 5 7 7 4 4
55)	1 3 5 7 8 4 4	61)	1 3 5 7 9 4 4	69)	1 3 6 4 4 4 4
70)	1 3 6 4 5 4 4	73)	1 3 6 5 4 4 4	74)	1 3 5 7 7 4 4
75)	1 3 6 5 5 5 4	76)	1 3 6 5 5 6 4	77)	1 3 6 5 6 4 4
81)	1 3 6 6 4 4 4	82)	1 3 6 6 5 4 4	83)	1 3 6 6 5 5 4
89)	1 3 6 6 7 4 4	94)	1 3 6 7 4 4 4	95)	1 3 6 7 5 4 4
96)	1 3 6 7 5 5 4	98)	1 3 6 7 6 4 4	100)	1 3 6 7 6 6 4
102)	1 3 6 7 7 4 4	107)	1 3 6 7 8 4 4	113)	1 3 6 7 9 4 4
121)	1 3 6 8 4 4 4	122)	1 3 6 8 5 4 4	123)	1 3 6 8 5 5 4
125)	1 3 6 8 6 4 4	127)	1 3 6 8 6 6 4	129)	1 3 6 8 7 4 4
132)	1 3 6 8 7 7 4	134)	1 3 6 8 8 4 4	140)	1 3 6 8 9 4 4
148)	1 3 6 8 10 4 4	157)	1 3 6 9 4 4 4	158)	1 3 6 9 5 4 4
159)	1 3 6 9 5 5 4	161)	1 3 6 9 6 4 4	163)	1 3 6 9 6 6 4
165)	1 3 6 9 7 4 4	168)	1 3 6 9 7 7 4	170)	1 3 6 9 8 4 4
174)	1 3 6 9 8 8 4	176)	1 3 6 9 9 4 4	184)	1 3 6 9 10 4 4
193)	1 3 6 9 11 4 4	202)	1 3 6 9 12 7 4	211)	1 3 6 10 4 4 4
212)	1 3 6 10 5 4 4	213)	1 3 6 10 5 5 4	215)	1 3 6 10 6 4 4
217)	1 3 6 10 6 6 4	219)	1 3 6 10 7 4 4	222)	1 3 6 10 7 7 4
224)	1 3 6 10 8 4 4	228)	1 3 6 10 8 8 4	230)	1 3 6 10 9 4 4
235)	1 3 6 10 9 9 4	238)	1 3 6 10 10 4 4	247)	1 3 6 10 11 4 4
256)	1 3 6 10 12 4 4	265)	1 3 6 10 13 4 4	274)	1 3 6 10 14 4 4
283)	1 3 6 10 15 4 4				

TABLE 2

Remark 2.10. Let $\mathbf{H}$ and R be as in Theorem 2.9. Note that any Artinian algebra $A = R/I$ with Hilbert function $\mathbf{H}$ has a socle element in degree $d-1$ or d.

Theorem 2.11 (Theorem 2.17, [3])**.** *Let $h_{d-2},\ h_{d-1},\ h_d$ be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d-1\rangle} \qquad \textit{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2\rangle}\ .$$

Let I be any ideal in $R = k[x_1, \dots, x_n]$ such that the Hilbert function of R/I satisfies

$$\begin{array}{lcl} \mathbf{H}(R/I, d-2) & = & h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) & = & h_{d-1}, \\ \mathbf{H}(R/I, d) & = & h_d\,. \end{array}$$

Then, the ring R/I has socle of dimension ε in degree $d-2$. Consequently, if I has minimal free resolution, then

$$\beta_{n-1,d+n-2} = \varepsilon.$$

Theorem 2.12 (Theorem 8, [8])**.** *If $\mathbf{H}\ :\ h_0\ \ h_1\ \ \cdots\ \ h_t\ \ 0\ \to$ is a level Artinian O-sequence where $t \geq 2$, then $\mathbf{G}\ :\ h_0\ \ h_1\ \ \cdots\ \ h_{t-1}\ \ 0\ \to$ is also a level Artinian O-sequence.*

Using Theorem 2.11, one can show that the following 100 h-vectors in Table 3 are not level O-sequences.

1)	1 3 3 4 4 4 4	2)	1 3 3 4 5 4 4	3)	1 3 3 4 5 5 4
4)	1 3 3 4 5 6 4	6)	1 3 4 4 5 4 4	7)	1 3 4 4 5 5 4
8)	1 3 4 4 5 6 4	10)	1 3 4 5 5 4 4	12)	1 3 4 5 5 6 4
13)	1 3 4 5 6 4 4	18)	1 3 5 4 5 4 4	19)	1 3 5 4 5 5 4
20)	1 3 5 4 5 6 4	22)	1 3 5 5 5 4 4	24)	1 3 5 5 5 6 4
25)	1 3 5 5 6 4 4	26)	1 3 5 5 6 5 4	27)	1 3 5 5 6 6 4
28)	1 3 5 5 6 7 4	30)	1 3 5 6 5 4 4	32)	1 3 5 6 5 6 4
33)	1 3 5 6 6 4 4	36)	1 3 5 6 6 7 4	37)	1 3 5 6 7 4 4
43)	1 3 5 7 5 4 4	45)	1 3 5 7 5 6 4	46)	1 3 5 7 6 4 4
49)	1 3 5 7 6 7 4	50)	1 3 5 7 7 4 4	54)	1 3 5 7 7 8 4
55)	1 3 5 7 8 4 4	61)	1 3 5 7 9 4 4	70)	1 3 6 4 5 4 4
71)	1 3 6 4 5 5 4	72)	1 3 6 4 5 6 4	74)	1 3 6 5 5 4 4
76)	1 3 6 5 5 6 4	77)	1 3 6 5 6 4 4	78)	1 3 6 5 6 5 4
79)	1 3 6 5 6 6 4	80)	1 3 6 5 6 7 4	82)	1 3 6 6 5 4 4
84)	1 3 6 6 5 6 4	85)	1 3 6 6 6 4 4	88)	1 3 6 6 6 7 4
89)	1 3 6 6 7 4 4	95)	1 3 6 7 5 4 4	97)	1 3 6 7 5 6 4
98)	1 3 6 7 6 4 4	101)	1 3 6 7 6 7 4	102)	1 3 6 7 7 4 4
106)	1 3 6 7 7 8 4	107)	1 3 6 7 8 4 4	113)	1 3 6 7 9 4 4
114)	1 3 6 7 9 5 4	115)	1 3 6 7 9 6 4	116)	1 3 6 7 9 7 4
117)	1 3 6 7 9 8 4	118)	1 3 6 7 9 9 4	119)	1 3 6 7 9 10 4
120)	1 3 6 7 9 11 4	122)	1 3 6 8 5 4 4	124)	1 3 6 8 5 6 4
125)	1 3 6 8 6 4 4	128)	1 3 6 8 6 7 4	129)	1 3 6 8 7 4 4
133)	1 3 6 8 7 8 4	134)	1 3 6 8 8 4 4	140)	1 3 6 8 9 4 4
147)	1 3 6 8 9 11 4	148)	1 3 6 8 10 4 4	158)	1 3 6 9 5 4 4
160)	1 3 6 9 5 6 4	161)	1 3 6 9 6 4 4	164)	1 3 6 9 6 7 4
165)	1 3 6 9 7 4 4	169)	1 3 6 9 7 8 4	170)	1 3 6 9 8 4 4
176)	1 3 6 9 9 4 4	183)	1 3 6 9 9 11 4	184)	1 3 6 9 10 4 4
192)	1 3 6 9 10 12 4	193)	1 3 6 9 11 4 4	202)	1 3 6 9 12 4 4
212)	1 3 6 10 5 4 4	214)	1 3 6 10 5 6 4	215)	1 3 6 10 6 4 4
218)	1 3 6 10 6 7 4	219)	1 3 6 10 7 4 4	223)	1 3 6 10 7 8 4
224)	1 3 6 10 8 4 4	230)	1 3 6 10 9 4 4	237)	1 3 6 10 9 11 4
238)	1 3 6 10 10 4 4	246)	1 3 6 10 10 12 4	247)	1 3 6 10 11 4 4
256)	1 3 6 10 12 4 4	265)	1 3 6 10 13 4 4	274)	1 3 6 10 14 4 4
283)	1 3 6 10 15 4 4				

TABLE 3

Remark 2.13 ([15])**.** Unfortunately, the linked sum construction cannot always be used to construct level Artinian algebras of type > 1 in codimension 3 from level sets of points in $\mathbb{P}^2$ (in sharp contrast to the case of Gorenstein algebras of codimension 3, see (Theorem 3.3, [6]).

An easy way to see this is to consider potential h-vectors describing Artinian algebras of socle degree 5 and type 4 (level). Let $h = (1, 3, -, -, \alpha, 4)$. By Inverse Systems we know that $\alpha \leq 12$ and, e.g. that there is a compressed level algebra with h-vector $(1, 3, 6, 10, 12, 4)$. However, that h-vector and indeed any h-vector of the form $(1, 3, -, -, \geq 11, 4)$ cannot be obtained from the linked-sum construction.

To see why, suppose we had a level set of points $\mathbb{Z} \subset \mathbb{P}^2$ and a partition $\mathbb{Z} = \mathbb{X} \bigcup \mathbb{Y}$ which gave the h-vector above by the linked-sum construction. Then, if we let $A = R/(I \quad + I \quad)$, we would have a display

$$\begin{array}{rcccccc}
\mathbf{H}(\mathbb{Z}, -): & 1 & - & - & - & b & c \\
\mathbf{H}(\mathbb{X}, -): & 1 & - & - & - & \alpha_1 & \alpha_2 \\
\mathbf{H}(\mathbb{Y}, -): & 1 & - & - & - & \beta_1 & \beta_2 \\
\mathbf{H}(A, -): & 1 & - & - & - & s & 4
\end{array}$$

where $11 \leq s \leq 12$.

The construction method says that $b+s = \alpha_1+\beta_1$ and $c+4 = \alpha_2+\beta_2$. Subtracting, we have

$$(c - b) + (4 - s) = (\alpha_2 - \alpha_1) + (\beta_2 - \beta_1).$$

But, since $\mathbb{X}$ and $\mathbb{Y}$ are point sets in $\mathbb{P}^2$ we have $(\alpha_2 - \alpha_1) \geq 0$ and $(\beta_2 - \beta_1) \geq 0$. Since $4 - s \leq -7$ we must have $c - b \geq 7$. But, since

$\mathbb{Z}$ has a differentiable Hilbert function, the maximum value for $c - b$ is $7 = \dim_k(k[x, y]_6)$.

The following lists are all those cases.

156)	1 3 6 8 10 12 4	192)	1 3 6 9 10 12 4	201)	1 3 6 9 11 12 4
210)	1 3 6 9 12 12 4	246)	1 3 6 10 10 12 4	255)	1 3 6 10 11 12 4
264)	1 3 6 10 12 12 4	273)	1 3 6 10 13 12 4	282)	1 3 6 10 14 12 4
291)	1 3 6 10 15 12 4				

TABLE 4

Proposition 2.14 (Theorem 4.8A, [3]). *Let $h = (1, h_1, \ldots, h_s)$ be the h-vector of a level algebra $A = R/I$ where $R = k[x_1, \ldots, x_n]$. Let $t < \binom{s+n-1}{s}$ and define the sequence*

$$(h + \mathbf{H}(s,t,n))_i := \min\left\{h_i + \mathbf{H}(s,t,n)_i, \binom{i+n-1}{i}\right\}.$$

Then $h + \mathbf{H}(s,t,n)$ is also the h-vector of a level algebra.

In [16], Yoo proved that the O-sequence 1 3 6 7 6 5 4 cannot be level using Proposition 2.14.

3. **Another cases of Non-level O-sequences**

Remark 3.1. Recall the Betti diagram of R/I:

$$
\begin{array}{c}
 \\ 0 \\ 1 \\ \vdots \\ d-3 \\ d-2 \\ d-1 \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{cccc} \quad & 0 & 1 & 2 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & * & * & * \\
\vdots & \vdots & \vdots & \vdots \\
0 & \beta_{0,d-2}(I) & \beta_{1,d-1}(I) & \beta_{2,d}(I) \\
0 & \beta_{0,d-1}(I) & \beta_{1,d}(I) & \beta_{2,d+1}(I) \\
0 & \beta_{0,d}(I) & \beta_{1,d+1}(I) & \beta_{2,d+2}(I) \\
\vdots & \vdots & \vdots &
\end{pmatrix}
\end{array}
$$

The goal of this section is to prove that some certain codimension 3 Artinian O-sequences **H** cannot be level. In other words, if R/I is a codimension 3 Artinian algebra of type 4 and socle degree 6 with Hilbert function **H** such that

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) + 1 \text{ or } \beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$$

where I^{lex} is a lex-segment ideal associated to I, then one can show that some of these cases cannot be level (see Propositions 3.3 and 3.6).

First of all, we introduce such cases in the following question.

Question 3.2. If I is the lex-segment ideal of $R = k[x_1, x_2, x_3]$ with Hilbert function $\mathbf{H}: 1 \quad 3 \quad 6 \quad 9 \quad 7 \quad 6 \quad 4$, then R/I has a minimal free resolution as follows.

$$\begin{array}{rl} 0 & \rightarrow \ \boldsymbol{R^3}(-\mathbf{6}) \oplus R^2(-7) \oplus R^2(-8) \oplus R^4(-9) \\ & \rightarrow \ R^8(-5) \oplus \boldsymbol{R^4}(-\mathbf{6}) \oplus R^5(-7) \oplus R^8(-8) \\ & \rightarrow \ R(-3) \oplus R^5(-4) \oplus R^2(-5) \oplus R^3(-6) \oplus R^4(-7) \\ & \rightarrow \ R \ \rightarrow \ R/I \ \rightarrow \ 0. \end{array} \tag{3.1}$$

Then the difference between the Betti number of the shift 6 in the last free module and the Betti number of the same shift in the middle free module is 1.

Now we shall prove that $\mathbf{H} : 1 \quad 3 \quad 6 \quad 9 \quad 7 \quad 6 \quad 4$ cannot be level in the following proposition.

Proposition 3.3. *The Artinian O-sequence* $\mathbf{H} : 1 \quad 3 \quad 6 \quad 9 \quad 7 \quad 6 \quad 4$ *cannot be level.*

Before we give the proof of this proposition, we recall the following theorem here.

Theorem 3.4 (Eliahou-Kervaire, [13])**.** *Let I be a stable monomial ideal of R. Denote by $\mathcal{G}(I)$ the set of minimal (monomial) generators of I and $\mathcal{G}(I)_d$ the elements of $\mathcal{G}(I)$ having degree d. Then*

$$\beta_{q,i}(I) = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T)-1}{q}$$

This theorem gives all the graded Betti numbers of the lex-segment ideal and the generic initial ideal just from an intimate knowledge of the generators of that ideal. Since the minimal free resolution of the ideal of a k-configuration in $\mathbb{P}^n$ is extremal ([4], [6]), we may apply this result to those ideals. It is an immediate consequence of the Eliahou–Kervaire theorem that if I is a lex-segment ideal, a generic initial ideal, or the ideal of a k-configuration in $\mathbb{P}^n$ which has *no* generators in degree d, then $\beta_{q,i} = 0$ whenever $i - q = d$.

Proof of Proposition 3.3. We assume that there exists an ideal I in $R = k[x_1, x_2, x_3]$ such that R/I is an Artinian level algebra with the given Hilbert function $\mathbf{H}$.

Let I^{lex} be a lex-segment ideal associated to I in $R = k[x_1, x_2, x_3]$. Then, by equation (3.1), the Betti diagram of R/I^{lex} is as follows:

```
total: 1   -   -   -
------------------------
    0: 1   -   -   -
    1: 0   0   0   0
    2: 0   1   0   0
    3: 0   5   8  (3)
    4: 0   2  (4)  2
    5: 0   3   5   2
```

```
      6: 0   4   8   4
```

In other words, we have

$$\beta_{1,6}(I^{\text{lex}}) = \beta_{2,6}(I^{\text{lex}}) + 1 = 4.$$

Let $K := (I_{\leq 5})^{\text{lex}}$. Note that I^{lex} and K agree in degree ≤ 5. Hence we can write the Betti diagram of R/K as

```
total: 1   -   -   -
------------------------
     0: 1   -   -   -
     1: 0   0   0   0
     2: 0   1   0   0
     3: 0   5   8  (3)
     4: 0   2  (4)  2
     5: 0   x   y   *
           .......
```

Since R/I is level and $(I_{\leq 5})$ has no generators in degree 6, we have

$$\beta_{0,6}((I_{\leq 5})) = \beta_{2,6}((I_{\leq 5})) = 0.$$

Moreover, by Lemma 2.9 (a) in [11], we see that

$$\begin{aligned} x &= \beta_{0,6}(K) \\ &= \beta_{1,6}(K) - \beta_{1,6}(I_{\le 5}) - \beta_{2,6}(K) \\ &\le \beta_{1,6}(K) - \beta_{2,6}(K) \\ &= 4 - 3 \\ &= 1, \end{aligned}$$

which follows that $x = 0$ or $x = 1$.

Case 1. Let $x = 0$. Then, by Theorem 3.4, we have $y = 0$. Moreover, by Lemma 2.9 (a) in [11] again, we have

$$\begin{aligned} \beta_{2,7}(K) - \beta_{2,7}((I_{\le 5})) &\le \beta_{1,7}(K) - \beta_{1,7}((I_{\le 5})) \\ &\le \beta_{1,7}(K) \\ &= y \\ &= 0, \end{aligned}$$

which follows that

$$\beta_{2,7}(K) = \beta_{2,7}((I_{\le 5})) = 2.$$

Hence $R/(I_{\le 5})$ has two dimensional socle elements in degree 4, so does R/I since $R/(I_{\le 5})$ and R/I agree in degree ≤ 5, which is a contradiction.

Case 2. If $x = 1$, then K has one generator in degree 6, and hence the Hilbert funtions of R/K and $R/(I_{\le 5})$ begin

$$1 \quad 3 \quad 6 \quad 9 \quad 7 \quad 6 \quad 6 \cdots .$$

Thus by Theorem 4.1 [11], $R/(I_{\le 5})$ has a socle element in degree 4, and so does R/I since $R/(I_{\le 5})$ and R/I agree in degree ≤ 5, which is a contradiction.

By Cases 1 and 2, the O-sequence $\mathbf{H}$ cannot be level, as we wished.

$\square$

Remark 3.5. Using the same idea as in the proof of Proposition 3.3, one can show that any graded Artinian algebra R/I with Hilbert function 1 3 6 10 8 7 4 cannot be level.

Here we prove that a slight different case from the above cannot be level, either.

Proposition 3.6. *The Artinian O-sequence* $\mathbf{H}: 1 \quad 3 \quad 6 \quad 10 \quad 10 \quad 6 \quad 4$ *cannot be level.*

Proof. We assume that there exists an ideal I in $R = k[x_1, x_2, x_3]$ such that R/I is an Artinian level algebra with the given Hilbert function $\mathbf{H}$.

Let I^{lex} be a lex-segment ideal associated to I in $R = k[x_1, x_2, x_3]$. Then the Betti diagram of R/I^{lex} is as follows:

```
total: 1   -   -   -
------------------------
    0: 1   -   -   -
```

```
   1: 0   0   0   0
   2: 0   0   0   0
   3: 0   5   6   2
   4: 0   6  11  (5)
   5: 0   3  (5)  2
   6: 0   4   8   4
```

In other words, we have

$$\beta_{1,7}(I^{\text{lex}}) = \beta_{2,7}(I^{\text{lex}}) = 5.$$

Let $K := (I_{\leq 6})^{\text{lex}}$. Note that I^{lex} and K agree in degree ≤ 6. Hence we can write the Betti diagram of R/K as

```
total: 1   -   -   -
------------------------
   0: 1   -   -   -
   1: 0   0   0   0
   2: 0   0   0   0
   3: 0   5   6   2
   4: 0   6  11  (5)
   5: 0   3  (5)  2
   6: 0   x   y   *
```

Since R/I is level and $(I_{\leq 6})$ has no generators in degree 7, we have

$$\beta_{0,7}((I_{\leq 6})) = \beta_{2,7}((I_{\leq 6})) = 0.$$

Moreover, by Lemma 2.9 (a) in [11], we see that

$$\begin{aligned} x &= \beta_{0,7}(K) \\ &= \beta_{1,7}(K) - \beta_{1,7}(I_{\leq 6}) - \beta_{2,7}(K) \\ &\leq \beta_{1,7}(K) - \beta_{2,7}(K) \\ &= 5 - 5 \\ &= 0, \end{aligned}$$

which follows that $x = 0$. Then, by Theorem 3.4, we have $y = 0$. Moreover, by Lemma 2.9 (a) in [11] again, we have

$$\begin{aligned} \beta_{2,8}(K) - \beta_{2,8}((I_{\leq 6})) &\leq \beta_{1,8}(K) - \beta_{1,8}((I_{\leq 6})) \\ &\leq \beta_{1,8}(K) \\ &= y \\ &= 0, \end{aligned}$$

which follows that

$$\beta_{2,8}(K) = \beta_{2,8}((I_{\leq 6})) = 2.$$

Hence $R/(I_{\leq 6})$ has two dimensional socle elements in degree 5, so does R/I since $R/(I_{\leq 6})$ and R/I agree in degree ≤ 6, which is a contradiction.

Therefore, R/I is not level, as we wished. $\square$

Remark 3.7. Using the same idea as in the proof of Proposition 3.6, one can show that the following cases in Table 5 cannot be level.

38) 1 3 5 6 7 5 4	51) 1 3 5 7 7 5 4	91) 1 3 6 6 7 6 4
103) 1 3 6 7 7 5 4	130) 1 3 6 8 7 5 4	150) 1 3 6 8 10 6 4
166) 1 3 6 9 7 5 4	186) 1 3 6 9 10 6 4	220) 1 3 6 10 7 5 4
268) 1 3 6 10 13 7 4		

TABLE 5

Morever, we can use the same idea to the cases $\beta_{1,6}(K) = \beta_{2,6}((I_{\leq 6}))$.

In other words, the following O-sequences in Table 6 cannot be level.

126) 1 3 6 8 6 5 4	221) 1 3 6 10 7 6 4	245) 1 3 6 10 10 11 4

TABLE 6

Appendix A.

The construction of some codimension 3 Level algebras of socle degree 6 and type 4

In this appendix, we introduce how to construct level O-sequences using the "link-sum" construction which are given by Cho in [1] and Yoo in [16]. They used the computer programs S-plus and CoCoA to produce them here.

Definition A.1. (Definition 2.1, [14])

(a) A finite set $\mathbb{X}$ of points $\mathbb{P}^2$ is called a ***basic configuration*** of type (d, e) if there exists distinct elements b_j, c_j in k such that

$$I = \left(\prod_{j=1}^{d}(x - b_j z), \prod_{j=1}^{e}(y - c_j z)\right).$$

We denote $\mathbb{X} := \mathbb{B}(d, e)$.

(b) A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a ***pure configuration*** if there exist finite basic configurations $\mathbb{B}(d_1, e_1), \ldots, \mathbb{B}(d_m, e_m)$ where $e_1 > \cdots > e_m$, which satisfy the following three conditions:

i) $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \phi$ if $i \neq j$,

ii) $\mathbb{X} = \mathbb{B}(d_1, e_1) \cup \cdots \cup \mathbb{B}(d_m, e_m)$,

iii) $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$ for all $1 \leq i \leq m-1$, where $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \to \mathbb{P}^1$ is the map defined by sending the point (x, y, z) to the point (y, z). In this case, we denote $\mathbb{X} = \cup_{i=1}^{m} \mathbb{B}(d_i, e_i)$.

Proposition A.2 (Proposition 3.8 in [5]). *Let $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ be a pure configuration in $\mathbb{P}^2$. Then a minimal free resolution of $\mathbb{X}$ is :*

$$0 \to \bigoplus_{i=1}^{m} R(-p_i) \to \bigoplus_{i=1}^{m+1} R(-q_i) \to R \to R/I \quad \to 0,$$

where

$$q_1 = e_1, \quad q_i = d_1 + \cdots + d_{i-1} + e_i \quad (2 \leqslant i \leqslant m),$$

$$q_{m+1} = d_1 + \cdots + d_m, \quad p_i = q_i + d_i \quad (1 \leqslant i \leqslant m).$$

Example A.3 (Example 3.9 in [5]). Let $\mathbb{X} = \mathbb{B}(3,6) \cup \mathbb{B}(4,2) \cup \mathbb{B}(2,2) \cup \mathbb{B}(1,1)$ be a pure configuration in $\mathbb{P}^2$.

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \right\}$$

The minimal free resolution of $\mathbb{Z}$ is

$0 \to R^4(-9) \to R(-6) \oplus R^2(-7) \oplus R^2(-8) \to R \to R/I \to 0.$

Corollary A.4 (Corollary 3.10, [5]). *Let* $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ *be a pure configuration in* $\mathbb{P}^2$. *Then* $\mathbb{X}$ *is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all $1 \leqslant i \leqslant m-1$.

If $r(A)$ is the Cohen-Macaulay type of a Cohen-Macaulay standard graded k-algebra A, and if $A = \bigoplus_{i \ \ 0} A_i$ is an Artinian level algebra, then $r(A) = dim_k A_{\sigma(A)-1}$, where $\sigma(A) = min\{i | A_i = 0\}$. If $\mathbb{Z} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ is a level pure configuration in $\mathbb{P}^2$, then $r(\mathbb{Z}) = r(R/I \ \ ) = m$.

Lemma A.5 (Lemma 3.14, [5]). *Let* $\mathbb{Z}$ *be a level set of points in* $\mathbb{P}^n$ *and* $\mathbb{X}$ *a subset of* $\mathbb{Z}$. *Set* $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. *Then* $R/(I \ \ + I \ \ )$ *is an Artinion level graded* k*-algebra with* $\sigma(R/(I \ \ + I \ \ )) = \sigma(\mathbb{Z}) - 1$ *and* $r(R/(I \ \ + I \ \ )) \leqslant r(\mathbb{Z})$.

Corollary A.6 (Corollary 3.15, [5]). *Let* $\mathbb{Z} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ *be a level pure configuration in* $\mathbb{P}^2$ *and* $\mathbb{X}$ *a subset of* $\mathbb{Z}$. *Set* $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. *Then* $R/(I \ \ + I \ \ )$ *is an Artinion level graded* k*-algebra with* $\sigma(R/(I \ \ + I \ \ )) = d_1 + e_1 - 2$ *and* $r(R/(I \ \ + I \ \ )) \leqslant m$.

Let $\mathbb{X}$ be the set of all $\bullet$'s in $\mathbb{Z}$ and $\mathbb{Y}$ be the set of all $*$'s in $\mathbb{Z}$. Next to each diagram we give the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $A = R/(I \ \ + I \ \ )$.

5)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & * & \bullet & & & \\ * & * & * & * & * & * & * & \\ * & \bullet & \bullet & * & * & * & * & * \end{array}\right\} \begin{array}{llrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 27 & \to \\ \mathbf{H}_A & : & 1 & 3 & 4 & 4 & 4 & 4 & 4 & 0 & \to . \end{array}$$

11)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & * & * & & & \\ * & * & * & * & \bullet & & & \\ * & * & * & * & * & * & * & \\ * & \bullet & \bullet & * & \bullet & * & \bullet & * \end{array}\right\} \begin{array}{llrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 26 & 26 & \to \\ \mathbf{H}_A & : & 1 & 3 & 4 & 5 & 5 & 5 & 4 & 0 & \to . \end{array}$$

15)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & * & \bullet & & & & & \\ * & * & \bullet & & & & & \\ * & * & \bullet & * & * & & & \\ * & * & \bullet & * & * & & & \\ * & * & \bullet & * & * & * & * & \\ * & * & * & \bullet & * & * & * & * \end{array}\right\} \begin{array}{llrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 25 & \to \\ \mathbf{H}_A & : & 1 & 3 & 4 & 5 & 6 & 6 & 4 & 0 & \to . \end{array}$$

16)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & \bullet & * & * & * & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & * & * & \\ \bullet & \bullet & * & * & \bullet & \bullet & \bullet & \bullet \end{array}\right\} \begin{array}{llrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 4 & 5 & 6 & 7 & 7 & 7 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 24 & 24 & \to \\ \mathbf{H}_A & : & 1 & 3 & 4 & 5 & 6 & 7 & 4 & 0 & \to . \end{array}$$

23)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ \bullet & * & \bullet & \bullet & * & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & * & * & * & * & \\ * & * & * & * & * & \bullet & * & * \end{array}\right\} \begin{array}{llrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 26 & 26 & \to \\ \mathbf{H}_A & : & 1 & 3 & 5 & 5 & 5 & 5 & 4 & 0 & \to . \end{array}$$

34)

$$\mathbb{Z} = \left\{\begin{array}{llllllll} * & * & * & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & * & * & * & \bullet & & & \\ * & * & * & * & * & & & \\ * & * & \bullet & * & * & * & * & \\ * & \bullet & * & * & * & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 25 & 25 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 6 & 6 & 5 & 4 & 0 & \rightarrow . \end{array}$$

35)

$$\mathbb{Z} = \left\{\begin{array}{llllllll} * & * & * & & & & & \\ * & \bullet & * & & & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & & & \\ \bullet & * & * & * & * & * & * & \\ \bullet & * & \bullet & * & \bullet & * & \bullet & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 25 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 6 & 6 & 6 & 4 & 0 & \rightarrow . \end{array}$$

39)

$$\mathbb{Z} = \left\{\begin{array}{llllllll} * & * & * & & & & & \\ * & * & * & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & * & * & \\ \bullet & \bullet & * & * & * & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 24 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 6 & 7 & 6 & 4 & 0 & \rightarrow . \end{array}$$

40)

$$\mathbb{Z} = \left\{\begin{array}{llllllll} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & \bullet & * & * & * & \\ \bullet & \bullet & * & \bullet & \bullet & \bullet & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 24 & 24 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 6 & 7 & 7 & 4 & 0 & \rightarrow . \end{array}$$

41)

$$\mathbb{Z} = \left\{\begin{array}{llllllll} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & * & * & & & \\ * & * & \bullet & * & * & & & \\ * & * & * & * & * & \bullet & * & \\ * & \bullet & \bullet & * & \bullet & \bullet & \bullet & \bullet \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 6 & 7 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 23 & 23 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 6 & 7 & 8 & 4 & 0 & \rightarrow . \end{array}$$

52)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \ast & \ast & \ast & & & & & \\ \ast & \ast & \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \\ \bullet & \ast & \ast & \bullet & \bullet & \ast & \bullet & \ast \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 7 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 24 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 7 & 7 & 6 & 4 & 0 & \rightarrow . \end{array}$$

53)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \ast & \bullet & \ast & & & & & \\ \ast & \bullet & \bullet & & & & & \\ \ast & \ast & \ast & \bullet & \ast & & & \\ \ast & \ast & \ast & \ast & \bullet & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast & \ast & \\ \ast & \bullet & \ast & \ast & \ast & \ast & \ast & \ast \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 7 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 24 & 24 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 7 & 7 & 7 & 4 & 0 & \rightarrow . \end{array}$$

58)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \ast & \ast & \ast & & & & & \\ \ast & \ast & \ast & & & & & \\ \ast & \ast & \ast & \bullet & \ast & & & \\ \ast & \ast & \ast & \bullet & \ast & & & \\ \bullet & \bullet & \ast & \bullet & \ast & \bullet & \bullet & \\ \ast & \ast & \ast & \bullet & \ast & \ast & \ast & \ast \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 7 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 23 & 23 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 7 & 8 & 7 & 4 & 0 & \rightarrow . \end{array}$$

59)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \ast & \ast & \ast & & & & & \\ \bullet & \ast & \ast & & & & & \\ \ast & \bullet & \ast & \ast & \ast & & & \\ \ast & \ast & \bullet & \ast & \ast & & & \\ \bullet & \bullet & \ast & \ast & \ast & \bullet & \bullet & \\ \ast & \ast & \ast & \ast & \bullet & \ast & \ast & \ast \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 7 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 23 & 23 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 7 & 8 & 8 & 4 & 0 & \rightarrow . \end{array}$$

60)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \ast & \ast & \bullet & & & & & \\ \ast & \ast & \ast & & & & & \\ \ast & \ast & \bullet & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \ast & & & \\ \ast & \ast & \bullet & \ast & \ast & \ast & \ast & \\ \ast & \bullet & \bullet & \bullet & \bullet & \ast & \bullet & \bullet \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 5 & 7 & 8 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 5 & 7 & 8 & 9 & 4 & 0 & \rightarrow . \end{array}$$

65)

$$\mathbb{Z} = \left\{\begin{array}{llllllll}
\ast & \ast & \ast \\
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\bullet & \bullet & \bullet & \bullet & \bullet \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\bullet & \ast & \ast & \bullet & \ast & \bullet & \bullet & \ast
\end{array}\right\}
\quad
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 5 & 7 & 9 & 9 & 9 & 9 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 22 & 22 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 5 & 7 & 9 & 8 & 4 & 0 & \rightarrow .
\end{array}$$

66)

$$\mathbb{Z} = \left\{\begin{array}{llllllll}
\ast & \bullet & \bullet \\
\ast & \ast & \ast \\
\ast & \ast & \bullet & \bullet & \ast \\
\ast & \ast & \bullet & \ast & \bullet \\
\ast & \ast & \bullet & \ast & \ast & \bullet & \ast \\
\ast & \ast & \bullet & \ast & \ast & \ast & \ast & \ast
\end{array}\right\}
\quad
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 5 & 7 & 9 & 9 & 9 & 9 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 22 & 22 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 5 & 7 & 9 & 9 & 4 & 0 & \rightarrow .
\end{array}$$

67)

$$\mathbb{Z} = \left\{\begin{array}{llllllll}
\ast & \ast & \bullet \\
\ast & \ast & \bullet \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \bullet & \ast & \ast \\
\ast & \ast & \bullet & \ast & \ast & \ast & \ast \\
\bullet & \bullet & \bullet & \bullet & \ast & \bullet & \bullet & \ast
\end{array}\right\}
\quad
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 5 & 7 & 9 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 21 & 21 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 5 & 7 & 9 & 10 & 4 & 0 & \rightarrow .
\end{array}$$

86)

$$\mathbb{Z} = \left\{\begin{array}{llllllll}
\bullet & \bullet & \bullet \\
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \bullet & \ast & \ast & \ast & \ast & \bullet \\
\bullet & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}\right\}
\quad
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 25 & 25 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 6 & 6 & 6 & 5 & 4 & 0 & \rightarrow .
\end{array}$$

87)

$$\mathbb{Z} = \left\{\begin{array}{llllllll}
\ast & \ast & \bullet \\
\ast & \ast & \ast \\
\bullet & \ast & \ast & \ast & \ast \\
\ast & \ast & \bullet & \bullet & \ast \\
\ast & \ast & \bullet & \ast & \ast & \ast & \ast \\
\bullet & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}\right\}
\quad
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 25 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 6 & 6 & 6 & 6 & 4 & 0 & \rightarrow .
\end{array}$$

104)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & \bullet & \bullet &&&&& \\ \bullet & \bullet & * &&&&& \\ * & * & * & * & * &&& \\ \bullet & * & * & * & * &&& \\ * & * & * & * & * & \bullet & * & \\ * & * & * & * & * & * & \bullet & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 24 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 7 & 7 & 6 & 4 & 0 & \rightarrow . \end{array}$$

105)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & \bullet & * &&&&& \\ * & * & * &&&&& \\ \bullet & \bullet & * & \bullet & * &&& \\ * & * & * & * & * &&& \\ * & * & * & \bullet & \bullet & * & * & \\ * & * & * & * & * & * & \bullet & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 24 & 24 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 7 & 7 & 7 & 4 & 0 & \rightarrow . \end{array}$$

110)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & \bullet & * &&&&& \\ * & * & * &&&&& \\ * & * & * & * & * &&& \\ \bullet & \bullet & \bullet & \bullet & \bullet &&& \\ * & * & * & * & \bullet & * & * & \\ * & * & * & * & * & * & * & \bullet \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 7 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 23 & 23 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 7 & 8 & 7 & 4 & 0 & \rightarrow . \end{array}$$

111)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & * & * &&&&& \\ * & * & * &&&&& \\ * & * & \bullet & * & * &&& \\ \bullet & * & * & * & * &&& \\ \bullet & * & \bullet & \bullet & * & \bullet & \bullet & \\ * & * & * & \bullet & * & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 7 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 23 & 23 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 7 & 8 & 8 & 4 & 0 & \rightarrow . \end{array}$$

112)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc} * & * & * &&&&& \\ * & \bullet & \bullet &&&&& \\ * & \bullet & * & * & * &&& \\ * & * & * & * & * &&& \\ * & * & * & * & * & * & * & \\ \bullet & \bullet & \bullet & \bullet & * & * & \bullet & \bullet \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 7 & 8 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 7 & 8 & 9 & 4 & 0 & \rightarrow . \end{array}$$

136)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}\bullet & \bullet & \bullet & & & & & \\ \bullet & * & \bullet & & & & & \\ * & * & * & * & * & & & \\ * & * & * & * & \bullet & & & \\ * & * & * & \bullet & * & * & * & \\ * & * & * & * & \bullet & * & * & *\end{array}\right\} \quad \begin{array}{lcrrrrrrrrl}\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 23 & 23 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 8 & 8 & 6 & 4 & 0 & \rightarrow .\end{array}$$

137)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}* & * & \bullet & & & & & \\ * & * & \bullet & & & & & \\ * & * & * & * & * & & & \\ * & * & \bullet & \bullet & \bullet & & & \\ * & * & \bullet & * & \bullet & * & * & \\ \bullet & * & * & * & * & * & * & *\end{array}\right\} \quad \begin{array}{lcrrrrrrrrl}\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 23 & 23 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 8 & 8 & 7 & 4 & 0 & \rightarrow .\end{array}$$

138)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}\bullet & * & * & & & & & \\ * & * & * & & & & & \\ * & * & \bullet & * & * & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & * & \bullet & \bullet & * & \\ * & \bullet & \bullet & * & * & * & * & \bullet\end{array}\right\} \quad \begin{array}{lcrrrrrrrrl}\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 23 & 10 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 8 & 8 & 8 & 4 & 0 & \rightarrow .\end{array}$$

143)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}\bullet & \bullet & \bullet & & & & & \\ \bullet & * & \bullet & & & & & \\ * & * & \bullet & * & * & & & \\ * & * & * & \bullet & * & & & \\ * & * & \bullet & * & * & * & * & \\ * & * & \bullet & * & * & * & * & *\end{array}\right\} \quad \begin{array}{lcrrrrrrrrl}\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 8 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 8 & 9 & 7 & 4 & 0 & \rightarrow .\end{array}$$

144)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}\bullet & * & * & & & & & \\ * & \bullet & * & & & & & \\ \bullet & * & \bullet & * & \bullet & & & \\ * & * & \bullet & \bullet & * & & & \\ * & * & * & * & * & \bullet & * & \\ * & * & * & * & * & \bullet & * & *\end{array}\right\} \quad \begin{array}{lcrrrrrrrrl}\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 8 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 8 & 9 & 8 & 4 & 0 & \rightarrow .\end{array}$$

145)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}
\ast & \bullet & \bullet &&&&& \\
\ast & \ast & \bullet &&&&& \\
\ast & \ast & \ast & \ast & \ast &&& \\
\ast & \bullet & \ast & \ast & \ast &&& \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \\
\ast & \ast & \bullet & \bullet & \bullet & \bullet & \ast & \bullet
\end{array}\right\}
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 8 & 9 & 9 & 9 & 9 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 22 & 22 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 6 & 8 & 9 & 9 & 4 & 0 & \rightarrow .
\end{array}$$

146)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}
\bullet & \ast & \ast &&&&& \\
\ast & \bullet & \ast &&&&& \\
\bullet & \ast & \bullet & \ast & \ast &&& \\
\ast & \ast & \ast & \bullet & \ast &&& \\
\ast & \ast & \ast & \ast & \bullet & \ast & \ast & \\
\bullet & \bullet & \ast & \bullet & \ast & \bullet & \ast & \ast
\end{array}\right\}
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 8 & 9 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 21 & 21 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 6 & 8 & 9 & 10 & 4 & 0 & \rightarrow .
\end{array}$$

152)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}
\bullet & \ast & \bullet &&&&& \\
\bullet & \ast & \bullet &&&&& \\
\bullet & \ast & \bullet & \ast & \ast &&& \\
\bullet & \ast & \bullet & \ast & \ast &&& \\
\bullet & \ast & \ast & \ast & \ast & \ast & \ast & \\
\ast & \ast & \ast & \ast & \ast & \ast & \bullet & \ast
\end{array}\right\}
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 8 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 21 & 21 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 6 & 8 & 10 & 8 & 4 & 0 & \rightarrow .
\end{array}$$

153)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}
\ast & \ast & \ast &&&&& \\
\ast & \ast & \ast &&&&& \\
\bullet & \bullet & \bullet & \bullet & \bullet &&& \\
\ast & \ast & \ast & \ast & \ast &&& \\
\ast & \ast & \ast & \ast & \ast & \bullet & \ast & \\
\ast & \bullet & \ast & \bullet & \bullet & \ast & \ast & \bullet
\end{array}\right\}
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 8 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 21 & 21 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 6 & 8 & 10 & 9 & 4 & 0 & \rightarrow .
\end{array}$$

154)

$$\mathbb{Z}=\left\{\begin{array}{cccccccc}
\ast & \bullet & \ast &&&&& \\
\ast & \bullet & \ast &&&&& \\
\ast & \bullet & \ast & \ast & \ast &&& \\
\ast & \ast & \ast & \ast & \ast &&& \\
\ast & \bullet & \bullet & \ast & \ast & \ast & \ast & \\
\bullet & \bullet & \ast & \bullet & \bullet & \bullet & \ast & \ast
\end{array}\right\}
\begin{array}{lcrrrrrrrrl}
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 8 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 21 & 21 & \rightarrow \\
\mathbf{H}_A & : & 1 & 3 & 6 & 8 & 10 & 10 & 4 & 0 & \rightarrow .
\end{array}$$

172)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & \bullet & * & * & * & & & \\ * & * & * & * & * & & & \\ * & * & \bullet & * & * & * & * & \\ \bullet & * & * & * & * & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 14 & 18 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 8 & 6 & 4 & 0 & \rightarrow . \end{array}$$

179)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} * & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & * & * & \bullet & * & & & \\ * & * & \bullet & * & * & & & \\ * & * & * & * & * & * & * & \\ * & \bullet & * & * & * & \bullet & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 9 & 7 & 4 & 0 & \rightarrow . \end{array}$$

180)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} * & * & * & & & & & \\ \bullet & * & \bullet & & & & & \\ * & \bullet & * & \bullet & \bullet & & & \\ * & * & * & \bullet & \bullet & & & \\ \bullet & * & * & * & * & * & * & \\ * & * & * & * & * & * & * & \bullet \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 9 & 8 & 4 & 0 & \rightarrow . \end{array}$$

181)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} * & \bullet & \bullet & & & & & \\ \bullet & * & * & & & & & \\ * & * & * & \bullet & * & & & \\ * & \bullet & * & * & * & & & \\ * & * & * & * & \bullet & \bullet & * & \\ * & * & * & * & \bullet & \bullet & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 22 & 22 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 9 & 9 & 4 & 0 & \rightarrow . \end{array}$$

187)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} * & \bullet & \bullet & & & & & \\ * & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & \bullet & * & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & * & * & * & * & \\ * & * & \bullet & * & * & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 21 & 21 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 10 & 7 & 4 & 0 & \rightarrow . \end{array}$$

188)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} * & * & \bullet & & & & & \\ * & * & \bullet & & & & & \\ * & * & \bullet & \bullet & * & & & \\ * & * & \bullet & * & \bullet & & & \\ * & * & \bullet & * & \bullet & \bullet & \bullet & \\ * & * & \bullet & * & * & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 10 & 11 & 11 & 11 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 20 & 20 & \to \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 10 & 8 & 4 & 0 & \to . \end{array}$$

189)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \bullet & * & * & & & & & \\ \bullet & * & * & & & & & \\ \bullet & * & * & * & \bullet & & & \\ \bullet & * & * & \bullet & * & & & \\ * & * & \bullet & \bullet & * & * & * & \\ * & \bullet & * & * & \bullet & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 10 & 10 & 10 & 10 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 21 & 21 & \to \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 10 & 9 & 4 & 0 & \to . \end{array}$$

190)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} * & * & \bullet & & & & & \\ * & \bullet & * & & & & & \\ \bullet & * & \bullet & * & * & & & \\ * & * & \bullet & * & * & & & \\ * & * & \bullet & * & * & * & * & \\ \bullet & \bullet & \bullet & * & \bullet & * & * & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 10 & 10 & 10 & 10 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 21 & 21 & \to \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 10 & 10 & 4 & 0 & \to . \end{array}$$

197)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & * & * & & & & & \\ \bullet & * & \bullet & * & * & & & \\ \bullet & * & \bullet & * & \bullet & & & \\ * & * & \bullet & * & * & * & * & \\ * & * & * & * & * & * & \bullet & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 11 & 11 & 11 & 11 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 20 & 20 & \to \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 11 & 8 & 4 & 0 & \to . \end{array}$$

198)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} * & * & \bullet & & & & & \\ * & * & * & & & & & \\ * & * & \bullet & * & * & & & \\ * & * & \bullet & * & * & & & \\ * & * & * & \bullet & \bullet & \bullet & \bullet & \\ * & * & \bullet & \bullet & \bullet & * & \bullet & * \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 11 & 11 & 11 & 11 & \to \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 20 & 20 & \to \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 11 & 9 & 4 & 0 & \to . \end{array}$$

199)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & * & & & & & \\ * & * & \bullet & & & & & \\ * & * & * & \bullet & * & & & \\ * & * & * & * & \bullet & & & \\ \bullet & \bullet & * & * & * & * & * & \\ * & * & \bullet & \bullet & \bullet & \bullet & * & \bullet \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 11 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 20 & 20 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 11 & 10 & 4 & 0 & \rightarrow . \end{array}$$

206)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & \bullet & & & & & \\ * & \bullet & * & & & & & \\ \bullet & * & * & \bullet & * & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ * & * & * & * & * & \bullet & * & \\ * & * & * & * & * & * & \bullet & * \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 12 & 12 & 12 & 12 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 17 & 19 & 19 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 12 & 8 & 4 & 0 & \rightarrow . \end{array}$$

207)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & * & * & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & \bullet & \bullet & \bullet & * & & & \\ * & * & \bullet & * & \bullet & & & \\ * & * & * & \bullet & * & \bullet & * & \\ * & * & * & * & * & * & \bullet & * \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 12 & 12 & 12 & 12 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 19 & 19 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 12 & 9 & 4 & 4 & \rightarrow . \end{array}$$

208)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & \bullet & & & & & \\ * & * & \bullet & & & & & \\ * & \bullet & \bullet & * & * & & & \\ * & \bullet & \bullet & * & * & & & \\ * & * & \bullet & * & * & * & * & \\ \bullet & \bullet & * & \bullet & * & * & * & \bullet \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 9 & 12 & 12 & 12 & 12 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 19 & 19 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 9 & 12 & 10 & 4 & 0 & \rightarrow . \end{array}$$

233)

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & \bullet & * & * & * & & & \\ * & * & * & \bullet & * & & & \\ * & \bullet & * & * & * & * & * & \\ \bullet & * & * & * & * & * & * & * \end{array} \right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 14 & 18 & 21 & 21 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 9 & 7 & 4 & 0 & \rightarrow . \end{array}$$

241)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \ast & \bullet & \bullet & & & & & \\ \ast & \bullet & \bullet & & & & & \\ \ast & \ast & \ast & \ast & \ast & & & \\ \bullet & \bullet & \bullet & \bullet & \ast & & & \\ \ast & \ast & \ast & \ast & \ast & \bullet & \ast & \\ \ast & \ast & \bullet & \ast & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 21 & 21 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 10 & 7 & 4 & 0 & \rightarrow . \end{array}$$

242)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \ast & \ast & \bullet & & & & & \\ \bullet & \bullet & \ast & & & & & \\ \bullet & \ast & \bullet & \ast & \ast & & & \\ \bullet & \bullet & \ast & \ast & \bullet & & & \\ \ast & \ast & \bullet & \ast & \ast & \ast & \ast & \\ \ast & \ast & \bullet & \ast & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 21 & 21 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 10 & 8 & 4 & 0 & \rightarrow . \end{array}$$

243)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \ast & \ast & \ast & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ \ast & \bullet & \ast & \ast & \ast & & & \\ \bullet & \ast & \ast & \ast & \bullet & & & \\ \ast & \bullet & \ast & \bullet & \ast & \ast & \ast & \\ \ast & \ast & \ast & \ast & \ast & \bullet & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 21 & 21 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 10 & 9 & 4 & 0 & \rightarrow . \end{array}$$

244)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \ast & \ast & \ast & & & & & \\ \ast & \ast & \bullet & & & & & \\ \ast & \ast & \ast & \ast & \bullet & & & \\ \ast & \ast & \bullet & \ast & \ast & & & \\ \ast & \bullet & \ast & \bullet & \ast & \bullet & \bullet & \\ \ast & \bullet & \ast & \bullet & \bullet & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 21 & 21 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 10 & 10 & 4 & 0 & \rightarrow . \end{array}$$

251)

$$\mathbb{Z} = \left\{\begin{array}{cccccccc} \ast & \ast & \ast & & & & & \\ \bullet & \ast & \ast & & & & & \\ \bullet & \bullet & \ast & \ast & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast & \ast & \\ \ast & \ast & \ast & \bullet & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 11 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 20 & 20 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 11 & 8 & 4 & 0 & \rightarrow . \end{array}$$

252)

$$\mathbb{Z} = \left\{\begin{matrix} \bullet & \bullet & * & & & & & \\ * & \bullet & \bullet & & & & & \\ * & * & \bullet & \bullet & \bullet & & & \\ * & * & \bullet & * & * & & & \\ * & * & * & * & \bullet & * & \bullet & \\ * & * & * & * & * & \bullet & * & * \end{matrix}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 11 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 19 & 20 & 20 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 11 & 9 & 4 & 0 & \rightarrow . \end{array}$$

253)

$$\mathbb{Z} = \left\{\begin{matrix} * & \bullet & \bullet & & & & & \\ \bullet & * & \bullet & & & & & \\ * & * & * & * & \bullet & & & \\ * & * & \bullet & * & * & & & \\ * & * & \bullet & * & \bullet & * & * & \\ * & * & \bullet & * & * & \bullet & * & \bullet \end{matrix}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 11 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 20 & 20 & 20 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 11 & 10 & 4 & 0 & \rightarrow . \end{array}$$

260)

$$\mathbb{Z} = \left\{\begin{matrix} \bullet & * & \bullet & & & & & \\ * & \bullet & \bullet & & & & & \\ * & * & * & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & * & \bullet & & & \\ * & * & * & \bullet & * & * & * & \\ * & * & * & * & * & * & * & \bullet \end{matrix}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 12 & 12 & 12 & 12 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 17 & 19 & 19 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 12 & 8 & 4 & 0 & \rightarrow . \end{array}$$

261)

$$\mathbb{Z} = \left\{\begin{matrix} \bullet & * & * & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & \bullet & \bullet & & & \\ * & \bullet & * & * & * & * & \bullet & \\ \bullet & * & * & * & * & \bullet & * & \bullet \end{matrix}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 12 & 12 & 12 & 12 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 19 & 19 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 12 & 9 & 4 & 0 & \rightarrow . \end{array}$$

262)

$$\mathbb{Z} = \left\{\begin{matrix} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & * & * & \bullet & * & & & \\ * & * & * & * & * & & & \\ * & \bullet & \bullet & \bullet & * & * & \bullet & \\ * & * & * & * & \bullet & * & * & \bullet \end{matrix}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 13 & 13 & 13 & 13 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 14 & 18 & 18 & 18 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 12 & 10 & 4 & 0 & \rightarrow . \end{array}$$

270)

$$\mathbb{Z}=\left\{\begin{array}{llllllll} \ast & \bullet & \ast \\ \ast & \ast & \bullet \\ \bullet & \bullet & \ast & \ast & \bullet \\ \bullet & \ast & \bullet & \ast & \ast \\ \bullet & \bullet & \bullet & \bullet & \ast & \ast & \bullet \\ \ast & \ast & \ast & \bullet & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 13 & 13 & 13 & 13 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 17 & 18 & 18 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 13 & 9 & 4 & 0 & \rightarrow . \end{array}$$

271)

$$\mathbb{Z}=\left\{\begin{array}{llllllll} \bullet & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \bullet & \bullet & \bullet \\ \ast & \bullet & \ast & \ast & \bullet \\ \ast & \bullet & \ast & \ast & \ast & \bullet & \bullet \\ \bullet & \ast & \ast & \bullet & \bullet & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 13 & 13 & 13 & 13 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 18 & 18 & 18 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 13 & 10 & 4 & 0 & \rightarrow . \end{array}$$

279)

$$\mathbb{Z}=\left\{\begin{array}{llllllll} \bullet & \bullet & \ast \\ \ast & \bullet & \bullet \\ \ast & \bullet & \ast & \bullet & \bullet \\ \ast & \bullet & \bullet & \ast & \bullet \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \bullet & \ast & \ast & \bullet & \bullet & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 14 & 14 & 14 & 14 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 16 & 17 & 17 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 14 & 9 & 4 & 0 & \rightarrow . \end{array}$$

280)

$$\mathbb{Z}=\left\{\begin{array}{llllllll} \ast & \bullet & \bullet \\ \ast & \bullet & \bullet \\ \ast & \ast & \ast & \ast & \ast \\ \bullet & \ast & \bullet & \ast & \bullet \\ \bullet & \bullet & \bullet & \ast & \ast & \bullet & \bullet \\ \ast & \ast & \ast & \ast & \bullet & \ast & \bullet & \ast \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 14 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 16 & 16 & 16 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 14 & 10 & 4 & 0 & \rightarrow . \end{array}$$

289)

$$\mathbb{Z}=\left\{\begin{array}{llllllll} \bullet & \ast & \bullet \\ \ast & \ast & \bullet \\ \ast & \ast & \bullet & \ast & \bullet \\ \bullet & \bullet & \ast & \ast & \bullet \\ \bullet & \ast & \ast & \ast & \bullet & \bullet & \bullet \\ \ast & \ast & \ast & \bullet & \ast & \bullet & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcrrrrrrrrl} \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 31 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H} & : & 1 & 3 & 6 & 10 & 15 & 16 & 16 & 16 & \rightarrow \\ \mathbf{H}_A & : & 1 & 3 & 6 & 10 & 15 & 10 & 4 & 0 & \rightarrow . \end{array}$$

Table 7. All possible Artinian O-sequences of codimension 3 with the length 7 and type 4.

1)	1 3 3 4 4 4 4	2)	1 3 3 4 5 4 4	3)	1 3 4 5 5 4 4
4)	1 3 3 4 5 6 4	5)	1 3 4 4 4 4 4	6)	1 3 4 4 5 4 4
7)	1 3 4 4 5 5 4	8)	1 3 4 4 5 6 4	9)	1 3 4 5 4 4 4
10)	1 3 4 5 5 4 4	11)	1 3 4 5 5 5 4	12)	1 3 4 5 5 6 4
13)	1 3 4 5 6 4 4	14)	1 3 4 5 6 5 4	15)	1 3 4 5 6 6 4
16)	1 3 4 5 6 7 4	17)	1 3 5 4 4 4 4	18)	1 3 5 4 5 4 4
19)	1 3 5 4 5 5 4	20)	1 3 5 4 5 6 4	21)	1 3 5 5 4 4 4
22)	1 3 5 5 5 4 4	23)	1 3 5 5 5 5 4	24)	1 3 5 5 5 6 4
25)	1 3 5 5 6 4 4	26)	1 3 5 5 6 5 4	27)	1 3 5 5 6 6 4
28)	1 3 5 5 6 7 4	29)	1 3 5 6 4 4 4	30)	1 3 5 6 5 4 4
31)	1 3 5 6 5 5 4	32)	1 3 5 6 5 6 4	33)	1 3 5 6 6 4 4
34)	1 3 5 6 6 5 4	35)	1 3 5 6 6 6 4	36)	1 3 5 6 6 7 4
37)	1 3 5 6 7 4 4	38)	1 3 5 6 7 5 4	39)	1 3 5 6 7 6 4
40)	1 3 5 6 7 7 4	41)	1 3 5 6 7 8 4	42)	1 3 5 7 4 4 4
43)	1 3 5 7 5 4 4	44)	1 3 5 7 5 5 4	45)	1 3 5 7 5 6 4
46)	1 3 5 7 6 4 4	47)	1 3 5 7 6 5 4	48)	1 3 5 7 6 6 4
49)	1 3 5 7 6 7 4	50)	1 3 5 7 7 4 4	51)	1 3 5 7 7 5 4
52)	1 3 5 7 7 6 4	53)	1 3 5 7 7 7 4	54)	1 3 5 7 7 8 4
55)	1 3 5 7 8 4 4	56)	1 3 5 7 8 5 4	57)	1 3 5 7 8 6 4
58)	1 3 5 7 8 7 4	59)	1 3 5 7 8 8 4	60)	1 3 5 7 8 9 4
61)	1 3 5 7 9 4 4	62)	1 3 5 7 9 5 4	63)	1 3 5 7 9 6 4
64)	1 3 5 7 9 7 4	65)	1 3 5 7 9 8 4	66)	1 3 5 7 9 9 4
67)	1 3 5 7 9 10 4	68)	1 3 5 7 9 11 4	69)	1 3 6 4 4 4 4
70)	1 3 6 4 5 4 4	71)	1 3 6 4 5 5 4	72)	1 3 6 4 5 6 4
73)	1 3 6 5 4 4 4	74)	1 3 6 5 5 4 4	75)	1 3 6 5 5 5 4
76)	1 3 6 5 5 6 4	77)	1 3 6 5 6 4 4	78)	1 3 6 5 6 5 4
79)	1 3 6 5 6 6 4	80)	1 3 6 5 6 7 4	81)	1 3 6 6 4 4 4
82)	1 3 6 6 5 4 4	83)	1 3 6 6 5 5 4	84)	1 3 6 6 5 6 4
85)	1 3 6 6 6 5 4	86)	1 3 6 6 6 5 4	87)	1 3 6 6 6 6 4
88)	1 3 6 6 6 7 4	89)	1 3 6 6 7 4 4	90)	1 3 6 6 7 5 4

91)	1 3 6 6 7 6 4	92)	1 3 6 6 7 7 4	93)	1 3 6 6 7 8 4
94)	1 3 6 7 4 4 4	95)	1 3 6 7 5 4 4	96)	1 3 6 7 5 5 4
97)	1 3 6 7 5 6 4	98)	1 3 6 7 6 4 4	99)	1 3 6 7 6 5 4
100)	1 3 6 7 6 6 4	101)	1 3 6 7 6 7 4	102)	1 3 6 7 7 4 4
103)	1 3 6 7 7 5 4	104)	1 3 6 7 7 6 4	105)	1 3 6 7 7 7 4
106)	1 3 6 7 7 8 4	107)	1 3 6 7 8 4 4	108)	1 3 6 7 8 5 4
109)	1 3 6 7 8 6 4	110)	1 3 6 7 8 7 4	111)	1 3 6 7 8 8 4
112)	1 3 6 7 8 9 4	113)	1 3 6 7 9 4 4	114)	1 3 6 7 9 5 4
115)	1 3 6 7 9 6 4	116)	1 3 6 7 9 7 4	117)	1 3 6 7 9 8 4
118)	1 3 6 7 9 9 4	119)	1 3 6 7 9 10 4	120)	1 3 6 7 9 11 4
121)	1 3 6 8 4 4 4	122)	1 3 6 8 5 4 4	123)	1 3 6 8 5 5 4
124)	1 3 6 8 5 6 4	125)	1 3 6 8 6 4 4	126)	1 3 6 8 6 5 4
127)	1 3 6 8 6 6 4	128)	1 3 6 8 6 7 4	129)	1 3 6 8 7 4 4
130)	1 3 6 8 7 5 4	131)	1 3 6 8 7 6 4	132)	1 3 6 8 7 7 4
133)	1 3 6 8 7 8 4	134)	1 3 6 8 8 4 4	135)	1 3 6 8 8 5 4
136)	1 3 6 8 8 6 4	137)	1 3 6 8 8 7 4	138)	1 3 6 8 8 8 4
139)	1 3 6 8 8 9 4	140)	1 3 6 8 9 4 4	141)	1 3 6 8 9 5 4
142)	1 3 6 8 9 6 4	143)	1 3 6 8 9 7 4	144)	1 3 6 8 9 8 4
145)	1 3 6 8 9 9 4	146)	1 3 6 8 9 10 4	147)	1 3 6 8 9 11 4
148)	1 3 6 8 10 4 4	149)	1 3 6 8 10 5 4	150)	1 3 6 8 10 6 4
151)	1 3 6 8 10 7 4	152)	1 3 6 8 10 8 4	153)	1 3 6 8 10 9 4
154)	1 3 6 8 10 10 4	155)	1 3 6 8 10 11 4	156)	1 3 6 8 10 12 4
157)	1 3 6 9 4 4 4	158)	1 3 6 9 5 4 4	159)	1 3 6 9 5 5 4
160)	1 3 6 9 5 6 4	161)	1 3 6 9 6 4 4	162)	1 3 6 9 6 5 4
163)	1 3 6 9 6 6 4	164)	1 3 6 9 6 7 4	165)	1 3 6 9 7 4 4
166)	1 3 6 9 7 5 4	167)	1 3 6 9 7 6 4	168)	1 3 6 9 7 7 4
169)	1 3 6 9 7 8 4	170)	1 3 6 9 8 4 4	171)	1 3 6 9 8 5 4
172)	1 3 6 9 8 6 4	173)	1 3 6 9 8 7 4	174)	1 3 6 9 8 8 4
175)	1 3 6 9 8 9 4	176)	1 3 6 9 9 4 4	177)	1 3 6 9 9 5 4
178)	1 3 6 9 9 6 4	179)	1 3 6 9 9 7 4	180)	1 3 6 9 9 8 4
181)	1 3 6 9 9 9 4	182)	1 3 6 9 9 10 4	183)	1 3 6 9 9 11 4
184)	1 3 6 9 10 4 4	185)	1 3 6 9 10 5 4	186)	1 3 6 9 10 6 4
187)	1 3 6 9 10 7 4	188)	1 3 6 9 10 8 4	189)	1 3 6 9 10 9 4
190)	1 3 6 9 10 10 4	191)	1 3 6 9 10 11 4	192)	1 3 6 9 10 12 4
193)	1 3 6 9 11 4 4	194)	1 3 6 9 11 5 4	195)	1 3 6 9 11 6 4
196)	1 3 6 9 11 7 4	197)	1 3 6 9 11 8 4	198)	1 3 6 9 11 9 4
199)	1 3 6 9 11 10 4	200)	1 3 6 9 11 11 4	201)	1 3 6 9 11 12 4

202)	1 3 6 9 12 4 4	203)	1 3 6 9 12 5 4	204)	1 3 6 9 12 6 4
205)	1 3 6 9 12 7 4	206)	1 3 6 9 12 8 4	207)	1 3 6 9 12 9 4
208)	1 3 6 9 12 10 4	209)	1 3 6 9 12 11 4	210)	1 3 6 9 12 12 4
211)	1 3 6 10 4 4 4	212)	1 3 6 10 5 4 4	213)	1 3 6 10 5 5 4
214)	1 3 6 10 5 6 4	215)	1 3 6 10 6 4 4	216)	1 3 6 10 6 5 4
217)	1 3 6 10 6 6 4	218)	1 3 6 10 6 7 4	219)	1 3 6 10 7 4 4
220)	1 3 6 10 7 5 4	221)	1 3 6 10 7 6 4	222)	1 3 6 10 7 7 4
223)	1 3 6 10 7 8 4	224)	1 3 6 10 8 4 4	225)	1 3 6 10 8 5 4
226)	1 3 6 10 8 6 4	227)	1 3 6 10 8 7 4	228)	1 3 6 10 8 8 4
229)	1 3 6 10 8 9 4	230)	1 3 6 10 9 4 4	231)	1 3 6 10 9 5 4
232)	1 3 6 10 9 6 4	233)	1 3 6 10 9 7 4	234)	1 3 6 10 9 8 4
235)	1 3 6 10 9 9 4	236)	1 3 6 10 9 10 4	237)	1 3 6 10 9 11 4
238)	1 3 6 10 10 4 4	239)	1 3 6 10 10 5 4	240)	1 3 6 10 10 6 4
241)	1 3 6 10 10 7 4	242)	1 3 6 10 10 8 4	243)	1 3 6 10 10 9 4
244)	1 3 6 10 10 10 4	245)	1 3 6 10 10 11 4	246)	1 3 6 10 10 12 4
247)	1 3 6 10 11 4 4	248)	1 3 6 10 11 5 4	249)	1 3 6 10 11 6 4
250)	1 3 6 10 11 7 4	251)	1 3 6 10 11 8 4	252)	1 3 6 10 11 9 4
253)	1 3 6 10 11 10 4	254)	1 3 6 10 11 11 4	255)	1 3 6 10 11 12 4
256)	1 3 6 10 12 4 4	257)	1 3 6 10 12 5 4	258)	1 3 6 10 12 6 4
259)	1 3 6 10 12 7 4	260)	1 3 6 10 12 8 4	261)	1 3 6 10 12 9 4
262)	1 3 6 10 12 10 4	263)	1 3 6 10 12 11 4	264)	1 3 6 10 12 12 4
265)	1 3 6 10 13 4 4	266)	1 3 6 10 13 5 4	267)	1 3 6 10 13 6 4
268)	1 3 6 10 13 7 4	269)	1 3 6 10 13 8 4	270)	1 3 6 10 13 9 4
271)	1 3 6 10 13 10 4	272)	1 3 6 10 13 11 4	273)	1 3 6 10 13 12 4
274)	1 3 6 10 14 4 4	275)	1 3 6 10 14 5 4	276)	1 3 6 10 14 6 4
277)	1 3 6 10 14 7 4	278)	1 3 6 10 14 8 4	279)	1 3 6 10 14 9 4
280)	1 3 6 10 14 10 4	281)	1 3 6 10 14 11 4	282)	1 3 6 10 14 12 4
283)	1 3 6 10 15 4 4	284)	1 3 6 10 15 5 4	285)	1 3 6 10 15 6 4
286)	1 3 6 10 15 7 4	287)	1 3 6 10 15 8 4	288)	1 3 6 10 15 9 4
289)	1 3 6 10 15 10 4	290)	1 3 6 10 15 11 4	291)	1 3 6 10 15 12 4

Appendix B.

Minimal Free Resolutions of The Rest of Unknown 30 Cases.

We give minimal free resolutions of unknown 30 cases which we should prove here.

14) 1 3 4 5 6 5 4

$$\begin{aligned} 0 &\rightarrow R(-7) \oplus R(-8) \oplus R^4(-9) \\ &\rightarrow R(-3) \oplus R^3(-6) \oplus R^2(-7) \oplus R^8(-8) \\ &\rightarrow R^2(-2) \oplus R^2(-5) \oplus R(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

57) 1 3 5 7 8 6 4

$$\begin{aligned} 0 &\rightarrow R^3(-7) \oplus R^2(-8) \oplus R^4(-9) \\ &\rightarrow R(-5) \oplus R^6(-6) \oplus R^5(-7) \oplus R^8(-8) \\ &\rightarrow R(-2) \oplus R(-4) \oplus R^3(-5) \oplus R^3(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

63) 1 3 5 7 9 6 4

$$\begin{aligned} 0 &\rightarrow R^4(-7) \oplus R^2(-8) \oplus R^4(-9) \\ &\rightarrow R^9(-6) \oplus R^5(-7) \oplus R^8(-8) \\ &\rightarrow R(-2) \oplus R^5(-5) \oplus R^3(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

64) 1 3 5 7 9 7 4

$$\begin{aligned} 0 &\rightarrow R^3(-7) \oplus R^3(-8) \oplus R^4(-9) \\ &\rightarrow R^7(-6) \oplus R^7(-7) \oplus R^8(-8) \\ &\rightarrow R(-2) \oplus R^4(-5) \oplus R^4(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

68) 1 3 5 7 9 11 4

$$\begin{aligned} 0 &\to R^7(-8) \oplus R^4(-9) \\ &\to R^{16}(-7) \oplus R^8(-8) \\ &\to R(-2) \oplus R^9(-6) \oplus R^4(-7) \\ &\to R \quad \to \quad R/I \quad \to \quad 0. \end{aligned}$$

109) 1 3 6 7 8 6 4

$$\begin{aligned} 0 &\to R(-5) \oplus R^3(-7) \oplus R^2(-8) \oplus R^4(-9) \\ &\to R^3(-4) \oplus R(-5) \oplus R^6(-6) \oplus R^5(-7) \oplus R^8(-8) \\ &\to R^3(-3) \oplus R(-4) \oplus R^3(-5) \oplus R^3(-6) \oplus R^4(-7) \\ &\to R \quad \to \quad R/I \quad \to \quad 0. \end{aligned}$$

131) 1 3 6 8 7 6 4

$$\begin{aligned} 0 &\to R^2(-6) \oplus R^2(-7) \oplus R^2(-8) \oplus R^4(-9) \\ &\to R(-4) \oplus R^5(-5) \oplus R^4(-6) \oplus R^5(-7) \oplus R^8(-8) \\ &\to R^2(-3) \oplus R^3(-4) \oplus R^2(-5) \oplus R^3(-6) \oplus R^4(-7) \\ &\to R \quad \to \quad R/I \quad \to \quad 0. \end{aligned}$$

142) 1 3 6 8 9 6 4

$$\begin{aligned} 0 &\to R(-6) \oplus R^4(-7) \oplus R^2(-8) \oplus R^4(-9) \\ &\to R(-4) \oplus R^2(-5) \oplus R^9(-6) \oplus R^5(-7) \oplus R^8(-8) \\ &\to R^2(-3) \oplus R(-4) \oplus R^5(-5) \oplus R^3(-6) \oplus R^4(-7) \\ &\to R \quad \to \quad R/I \quad \to \quad 0. \end{aligned}$$

151) 1 3 6 8 10 7 4

$$\begin{aligned} 0 &\to R^4(-7) \oplus R^3(-8) \oplus R^4(-9) \\ &\to R(-4) \oplus R^9(-6) \oplus R^7(-7) \oplus R^8(-8) \\ &\to R^2(-3) \oplus R^5(-5) \oplus R^4(-6) \oplus R^4(-7) \\ &\to R \quad \to \quad R/I \quad \to \quad 0. \end{aligned}$$

155) 1 3 6 8 10 11 4

$$\begin{array}{rl} 0 & \rightarrow \quad R(-7) \oplus R^7(-8) \oplus R^4(-9) \\ & \rightarrow \quad R(-4) \oplus R^2(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\ & \rightarrow \quad R^2(-3) \oplus R(-5) \oplus R^9(-6) \oplus R^4(-7) \\ & \rightarrow \quad R \quad \rightarrow \quad R/I \quad \rightarrow \quad 0. \end{array}$$

173) 1 3 6 9 8 7 4

$$\begin{array}{rl} 0 & \rightarrow \quad R^2(-6) \oplus R^2(-7) \oplus R^3(-8) \oplus R^4(-9) \\ & \rightarrow \quad R^6(-5) \oplus R^4(-6) \oplus R^7(-7) \oplus R^8(-8) \\ & \rightarrow \quad R(-3) \oplus R^4(-4) \oplus R^2(-5) \oplus R^4(-6) \oplus R^4(-7) \\ & \rightarrow \quad R \quad \rightarrow \quad R/I \quad \rightarrow \quad 0. \end{array}$$

178) 1 3 6 9 9 6 4

$$\begin{array}{rl} 0 & \rightarrow \quad R^2(-6) \oplus R^4(-7) \oplus R^2(-8) \oplus R^4(-9) \\ & \rightarrow \quad R^5(-5) \oplus R^9(-6) \oplus R^5(-7) \oplus R^8(-8) \\ & \rightarrow \quad R(-3) \oplus R^3(-4) \oplus R^5(-5) \oplus R^3(-6) \oplus R^4(-7) \\ & \rightarrow \quad R \quad \rightarrow \quad R/I \quad \rightarrow \quad 0. \end{array}$$

196) 1 3 6 9 11 7 4

$$\begin{array}{rl} 0 & \rightarrow \quad R^5(-7) \oplus R^3(-8) \oplus R^4(-9) \\ & \rightarrow \quad R(-5) \oplus R^{11}(-6) \oplus R^7(-7) \oplus R^8(-8) \\ & \rightarrow \quad R(-3) \oplus R(-4) \oplus R^6(-5) \oplus R^4(-6) \oplus R^4(-7) \\ & \rightarrow \quad R \quad \rightarrow \quad R/I \quad \rightarrow \quad 0. \end{array}$$

200) 1 3 6 9 11 11 4

$$\begin{array}{rl} 0 & \rightarrow \quad R^2(-7) \oplus R^7(-8) \oplus R^4(-9) \\ & \rightarrow \quad R(-5) \oplus R^4(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\ & \rightarrow \quad R(-3) \oplus R(-4) \oplus R^2(-5) \oplus R^9(-6) \oplus R^4(-7) \\ & \rightarrow \quad R \quad \rightarrow \quad R/I \quad \rightarrow \quad 0. \end{array}$$

205) 1 3 6 9 12 7 4

$$\begin{aligned} 0 &\rightarrow R^6(-7) \oplus R^3(-8) \oplus R^4(-9) \\ &\rightarrow R^{14}(-6) \oplus R^7(-7) \oplus R^8(-8) \\ &\rightarrow R(-3) \oplus R^8(-5) \oplus R^4(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

209) 1 3 6 9 12 11 4

$$\begin{aligned} 0 &\rightarrow R^3(-7) \oplus R^7(-8) \oplus R^4(-9) \\ &\rightarrow R^7(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\ &\rightarrow R(-3) \oplus R^4(-5) \oplus R^9(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

226) 1 3 6 10 8 6 4

$$\begin{aligned} 0 &\rightarrow R^3(-6) \oplus R^3(-7) \oplus R^2(-8) \oplus R^4(-9) \\ &\rightarrow R^9(-5) \oplus R^6(-6) \oplus R^5(-7) \oplus R^8(-8) \\ &\rightarrow R^7(-4) \oplus R^3(-5) \oplus R^3(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

232) 1 3 6 10 9 6 4

$$\begin{aligned} 0 &\rightarrow R^3(-6) \oplus R^4(-7) \oplus R^2(-8) \oplus R^4(-9) \\ &\rightarrow R^8(-5) \oplus R^9(-6) \oplus R^5(-7) \oplus R^8(-8) \\ &\rightarrow R^6(-4) \oplus R^5(-5) \oplus R^3(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

234) 1 3 6 10 9 8 4

$$\begin{aligned} 0 &\rightarrow R^3(-6) \oplus R^2(-7) \oplus R^4(-8) \oplus R^4(-9) \\ &\rightarrow R^8(-5) \oplus R^5(-6) \oplus R^9(-7) \oplus R^8(-8) \\ &\rightarrow R^6(-4) \oplus R^3(-5) \oplus R^5(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

250) 1 3 6 10 11 7 4

$$\begin{aligned} 0 &\to R(-6) \oplus R^5(-7) \oplus R^3(-8) \oplus R^4(-9) \\ &\to R^4(-5) \oplus R^{11}(-6) \oplus R^7(-7) \oplus R^8(-8) \\ &\to R^4(-4) \oplus R^6(-5) \oplus R^4(-6) \oplus R^4(-7) \\ &\to R \to R/I \to 0. \end{aligned}$$

254) 1 3 6 10 11 11 4

$$\begin{aligned} 0 &\to R(-6) \oplus R^2(-7) \oplus R^7(-8) \oplus R^4(-9) \\ &\to R^4(-5) \oplus R^4(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\ &\to R^4(-4) \oplus R^2(-5) \oplus R^9(-6) \oplus R^4(-7) \\ &\to R \to R/I \to 0. \end{aligned}$$

259) 1 3 6 10 12 7 4

$$\begin{aligned} 0 &\to R(-6) \oplus R^6(-7) \oplus R^3(-8) \oplus R^4(-9) \\ &\to R^3(-5) \oplus R^{14}(-6) \oplus R^7(-7) \oplus R^8(-8) \\ &\to R^3(-4) \oplus R^8(-5) \oplus R^4(-6) \oplus R^4(-7) \\ &\to R \to R/I \to 0. \end{aligned}$$

263) 1 3 6 10 12 11 4

$$\begin{aligned} 0 &\to R(-6) \oplus R^3(-7) \oplus R^7(-8) \oplus R^4(-9) \\ &\to R^3(-5) \oplus R^7(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\ &\to R^3(-4) \oplus R^4(-5) \oplus R^9(-6) \oplus R^4(-7) \\ &\to R \to R/I \to 0. \end{aligned}$$

269) 1 3 6 10 13 8 4

$$\begin{aligned} 0 &\to R^6(-7) \oplus R^4(-8) \oplus R^4(-9) \\ &\to R(-5) \oplus R^{14}(-6) \oplus R^9(-7) \oplus R^8(-8) \\ &\to R^2(-4) \oplus R^8(-5) \oplus R^5(-6) \oplus R^4(-7) \\ &\to R \to R/I \to 0. \end{aligned}$$

272) 1 3 6 10 13 11 4

$$\begin{aligned} 0 &\rightarrow R^4(-7) \oplus R^7(-8) \oplus R^4(-9) \\ &\rightarrow R(-5) \oplus R^9(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\ &\rightarrow R^2(-4) \oplus R^5(-5) \oplus R^9(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

278) 1 3 6 10 14 8 4

$$\begin{aligned} 0 &\rightarrow R^7(-7) \oplus R^4(-8) \oplus R^4(-9) \\ &\rightarrow R^{17}(-6) \oplus R^9(-7) \oplus R^8(-8) \\ &\rightarrow R(-4) \oplus R^{10}(-5) \oplus R^5(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

281) 1 3 6 10 14 11 4

$$\begin{aligned} 0 &\rightarrow R^5(-7) \oplus R^7(-8) \oplus R^4(-9) \\ &\rightarrow R^{12}(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\ &\rightarrow R(-4) \oplus R^7(-5) \oplus R^9(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

287) 1 3 6 10 15 8 4

$$\begin{aligned} 0 &\rightarrow R^8(-7) \oplus R^4(-8) \oplus R^4(-9) \\ &\rightarrow R^{20}(-6) \oplus R^9(-7) \oplus R^8(-8) \\ &\rightarrow R^{13}(-5) \oplus R^5(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

288) 1 3 6 10 15 9 4

$$\begin{aligned} 0 &\rightarrow R^7(-7) \oplus R^5(-8) \oplus R^4(-9) \\ &\rightarrow R^{18}(-6) \oplus R^{11}(-7) \oplus R^8(-8) \\ &\rightarrow R^{12}(-5) \oplus R^6(-6) \oplus R^4(-7) \\ &\rightarrow R \rightarrow R/I \rightarrow 0. \end{aligned}$$

290) 1 3 6 10 15 11 4

$$
\begin{aligned}
0 &\to R^6(-7) \oplus R^7(-8) \oplus R^4(-9) \\
&\to R^{15}(-6) \oplus R^{16}(-7) \oplus R^8(-8) \\
&\to R^{10}(-5) \oplus R^9(-6) \oplus R^4(-7) \\
&\to R \quad \to \quad R/I \quad \to \quad 0.
\end{aligned}
$$

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