

申 龍 秀 教授指導
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Non Existence of Artinian Level
Algebra of codimension 3 and
type 3

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誠信女子大學校 教育大學院

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認 准 書

金智惠의 碩士學位 論文을 認准함.

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논문개요

[1]에서 코코아 알고리즘을 이용하고 Fröberg and Laksov 의 정리에 근거하여 여차원이 3, 형태가 3, 길이가 7인 level이 될 가능성이 있는 334개의 Artinian O-수열에 대하여 level 여부를 연구하였다.

레벨이 되지 않는 수열은 증명을 통하여 설명하고 있고, 레벨인지 아닌지 알 수 없는 68개의 수열이 남았다. 마지막으로 s-plus와 코코아 알고리즘을 사용하여 레벨이 되는 수열은 'Link-sum' 구성을 하여 소개하고 있다.

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Abstract	

1. Introduction

A Gorenstein algebra has been much studied, so we are interested in level Artinian O-sequences. There have been many results on type 2 of level Artinian O-sequences of codimension 3 with length 7 (see [9], [14]), but not much on type 3 with length 7 since the type 3 Artinian O-sequences are more complicated.

Our main interest is to examine what kinds of Artinian O-sequences of type 3 can be level or not. We attempt to prove in two ways that codimension 3 Artinian O-sequences of type 3 and length 7 cannot be level. One is for the cases which need long proofs and the other is what can be proved by using different ideas.

It seems to be wise to start with the definition of Hilbert function. If we let $R = k[x_0, x_1, \dots, x_n] = \bigoplus_{i \geq 0} R_i$, where k is an algebraically closed field of characteristic 0, and let I be a homogeneous ideal of R , $A = R/I$, then the **Hilbert function** of A , $\mathbf{H}_A : \mathbb{N} \rightarrow \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R . The ***h-vector*** of A is $h(A) = (h_0, h_1, \dots, h_\ell)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and ℓ is the last index such that $\dim_k A_k \neq 0$. We call ℓ the ***socle degree*** A . Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the ***codimension*** of A .

Using the algorithm in [13] from CoCoA [12] and a special case of a theorem of Fröberg and Laksov in [7], we have figured out that there are 334 Artinian O-sequences which could be level (See Appendix A). To investigate these cases further, we consider one of those sequences, $1 \ 3 \ 5 \ 4 \ 4 \ 4 \ 3$. Let I be a lex-segment ideal of $R = k[x, y, z]$ with the Hilbert function $1 \ 3 \ 5 \ 4 \ 4$. Then the minimal free resolution of R/I is

$$\begin{aligned} 0 &\rightarrow R^2(-5) \oplus R^4(-7) \rightarrow R^5(-4) \oplus R(-5) \oplus R^8(-6) \\ &\rightarrow R(-2) \oplus R^3(-3) \oplus R(-4) \oplus R^4(-5) \rightarrow R \rightarrow 0. \end{aligned}$$

This indicates that both copies $R(-5)$ of the last free module cannot be canceled. This means an Artinian algebra with Hilbert function having a sequence, $1 \ 3 \ 5 \ 4 \ 4$, cannot be level, and thus, by Theorem 8 in [13], any Artinian algebra with Hilbert function having the sequence, $1 \ 3 \ 5 \ 4 \ 4 \ 4 \ 3$, cannot be level, either.

We try to classify Artinian O-sequences of codimension 3 and type 3 with length 7 based on whether or not being level. In this thesis, we shall prove that some O-sequences among the above 334 cases, which look like possible level Artinian O-sequences, cannot be level Artinian O-sequences and produce some of them using ‘Link-sum’ constructions of sets of points in \mathbb{P}^2 .

2. Non-Level Artinian O-sequences of codimension 3 and type 3

First of all, we introduce some definitions here.

Definition 2.1. (a) Let $h, i > 0$.

Then $h = \binom{m_i}{i} + \binom{m_i - 1}{i - 1} + \cdots + \binom{m_j}{j}$ with $m_i > m_i - 1 > \cdots > m_j \geq 1$

is called ***i-th binomial expansion***.

(b) The sequence $\{h_i\}_{i \geq 0}$ ($h_i \geq 0$) is called an ***O-Sequence*** if

$$h_{i+1} \leq h_i^{\langle i \rangle}, \quad h_0 = 1, \quad i \geq 1.$$

Definition 2.2. Let $A = \bigoplus_{i \geq 0} A_i$ be graded ring. Then $A = k[A_1]$ is called ***G-algebra*** if A_1 is a finitely generated vector space over k .

Theorem 2.3 (Macaulay). (a) $\{h_i\}_{i \geq 0}$ is an *O-Sequence*.

(b) $\{h_i\}_{i \geq 0}$ is the *Hilbert function* for some standard *G-algebra*.

Definition-Proposition 2.4 (Definition-Proposition 2.21 in [10]). Let $R = k[x_0, \dots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d . Let

$$0 \rightarrow \mathcal{F}_{n-(d-1)} \rightarrow \cdots \rightarrow \mathcal{F}_1 \rightarrow I \rightarrow 0$$

be a minimal free resolution of I . Then

- (a) If $B = B_0 \oplus \cdots \oplus B_\ell$ ($B_\ell \neq 0$) is an Artinian algebra, then B is **level** if and only if $B_\ell = \text{Ann}(B_1)$.
- (b) A is a **level algebra** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$.
 $\text{rank } \mathcal{F}_{n-(d-1)}$
 $=$ Cohen-Macaulay type of A .
- (c) i) If \mathbb{X} is a non-degenerate set of points in \mathbb{P}^n , $A = R/I_{\mathbb{X}}$ its coordinate ring, then we say that ℓ is the **socle degree** of \mathbb{X} if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where \overline{L} is any linear non-zero-divisor of A .
 ii) \mathbb{X} is called a **level set** of points if $A = R/I_{\mathbb{X}}$ is a level algebra. In this case, the socle degree of \mathbb{X} is $\ell = \sigma(\mathbb{X}) + n - 1$.
- (d) If \overline{L} is a linear non-zero divisor in $A = R/I$, then A is level if and only if $A/\overline{L}A \simeq A/(L, I_{\mathbb{X}})$ is level.
- (e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an Artinian algebra) $b = \{b_i\}_{i \geq 0}$ with $b_1 = n + 1$, is called **level** if there is a level set of points in with Hilbert function b .

Definition 2.5. (a) A total order on the monomials of each degree is said to be a **term order** if

- i) $x_1 > \cdots > x_n$, and

- ii) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m, m_1 and m_2 .
- (b) The **reverse lexicographic order** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if
- i) $\sum i_t > \sum j_t$ or
 - ii) $\sum i_t = \sum j_t$ and
there is s such that $i_t = j_t$ for $s < t \leq n$ and $i_s < j_s$.
- (c) The **lexicographic order** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if
- i) $\sum i_t > \sum j_t$ or
 - ii) $\sum i_t = \sum j_t$ and
there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.
- (d) Let S be a subset of all monomials in R_d . S is a **lex-segment** if a monomial m of degree d is in S , then every monomial m' of degree d in R_d such that $m' > m$ is in S .
- (e) Let $I = \bigoplus_{t \geq 0} I_t$ be a graded ideal of R . We say that I is a **lex-segment ideal** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

Theorem 2.6 (The Cancellation Principle, [1], [11]). *For any homogeneous ideal I and any i and d , there a complex of $k \cong R/m$ – modules V_\bullet^d such that*

$$\begin{aligned} V_i^d &\cong \operatorname{Tor}_i^R(\operatorname{in}(I), k)_d \\ H_i(V_\bullet^d) &\cong \operatorname{Tor}_i^R(I, k)_d. \end{aligned}$$

Remark 2.7. One way to paraphrase this theorem is to say that the minimal free resolution of I is obtained from that $\operatorname{in}(I)$, the *initial ideal* of I , by canceling some adjacent terms of the same degree.

The following 41 cases in Table 1 are not level by Theorem 2.6 and Remark 2.7.

62)	1 3 5 6 7 5 3	78)	1 3 5 7 6 7 3	86)	1 3 5 7 8 4 3
93)	1 3 5 7 9 4 3	133)	1 3 6 6 7 5 3	134)	1 3 6 6 7 6 3
135)	1 3 6 6 7 7 3	136)	1 3 6 6 7 8 3	164)	1 3 6 7 9 5 3
165)	1 3 6 7 9 6 3	166)	1 3 6 7 9 7 3	167)	1 3 6 7 9 8 3
168)	1 3 6 7 9 9 3	173)	1 3 6 8 5 4 3	193)	1 3 6 8 8 9 3
202)	1 3 6 8 10 4 3	203)	1 3 6 8 10 5 3	212)	1 3 6 9 5 4 3
217)	1 3 6 9 6 5 3	232)	1 3 6 9 8 9 3	242)	1 3 6 9 10 5 3
249)	1 3 6 9 11 5 3	255)	1 3 6 9 12 4 3	256)	1 3 6 9 12 5 3
265)	1 3 6 10 5 4 3	270)	1 3 6 10 6 5 3	272)	1 3 6 10 6 7 3
278)	1 3 6 10 7 8 3	285)	1 3 6 10 8 9 3	295)	1 3 6 10 10 5 3
302)	1 3 6 10 11 5 3	309)	1 3 6 10 12 5 3	315)	1 3 6 10 13 4 3
316)	1 3 6 10 13 5 3	317)	1 3 6 10 13 6 3	322)	1 3 6 10 14 4 3
323)	1 3 6 10 14 5 3	324)	1 3 6 10 14 6 3	329)	1 3 6 10 15 4 3
330)	1 3 6 10 15 5 3	331)	1 3 6 10 13 6 3		

TABLE 1

Theorem 2.8 ([2]). *Let $R = k[x_1, x_2, x_3]$ and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the h -vector of a graded Artinian algebra $A = R/I$ with socle degree s . If*

$$h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \leq 2d + 3$$

then \mathbf{H} is not level.

Remark 2.9. Let \mathbf{H} and R be as above. Then any Artinian algebra $A = R/I$ with Hilbert function \mathbf{H} has a socle element in degree $d - 1$.

The following 85 cases in Table 2 are not level by Theorem 2.8.

2)	1 3 3 4 3 3 3	3)	1 3 3 4 4 3 3	5)	1 3 3 4 5 3 3
9)	1 3 4 3 3 3 3	10)	1 3 4 4 3 3 3	11)	1 3 4 4 4 3 3
13)	1 3 4 4 5 3 3	17)	1 3 4 5 3 3 3	19)	1 3 4 5 4 4 3
20)	1 3 4 5 5 3 3	24)	1 3 4 5 6 3 3	29)	1 3 5 3 3 3 3
30)	1 3 5 4 3 3 3	31)	1 3 5 4 4 3 3	32)	1 3 5 4 4 4 3
37)	1 3 5 5 3 3 3	38)	1 3 5 5 4 3 3	39)	1 3 5 5 4 4 3
40)	1 3 5 5 5 3 3	44)	1 3 5 5 6 3 3	49)	1 3 5 6 3 3 3
51)	1 3 5 6 4 4 3	54)	1 3 5 6 5 5 3	56)	1 3 5 6 6 3 3
61)	1 3 5 6 7 3 3	67)	1 3 5 7 3 3 3	69)	1 3 5 7 4 4 3
72)	1 3 5 7 5 5 3	77)	1 3 5 7 6 6 3	79)	1 3 5 7 7 3 3
85)	1 3 5 7 8 3 3	92)	1 3 5 7 9 3 3	99)	1 3 6 3 3 3 3
101)	1 3 6 4 4 3 3	102)	1 3 6 4 4 4 3	109)	1 3 6 5 4 4 3
110)	1 3 6 5 5 3 3	111)	1 3 6 5 5 4 3	112)	1 3 6 5 5 5 3
113)	1 3 6 5 5 6 3	119)	1 3 6 6 3 3 3	120)	1 3 6 6 4 3 3
121)	1 3 6 6 4 4 3	122)	1 3 6 6 5 3 3	124)	1 3 6 6 5 5 3
126)	1 3 6 6 6 3 3	131)	1 3 6 6 7 3 3	137)	1 3 6 7 3 3 3
139)	1 3 6 7 4 4 3	142)	1 3 6 7 5 5 3	147)	1 3 6 7 6 6 3
149)	1 3 6 7 7 3 3	155)	1 3 6 7 8 3 3	162)	1 3 6 7 9 3 3
169)	1 3 6 8 3 3 3	171)	1 3 6 8 4 4 3	174)	1 3 6 8 5 5 3
179)	1 3 6 8 6 6 3	185)	1 3 6 8 7 7 3	187)	1 3 6 8 8 3 3
194)	1 3 6 8 9 3 3	201)	1 3 6 8 10 3 3	208)	1 3 6 9 3 3 3
210)	1 3 6 9 4 4 3	213)	1 3 6 9 5 5 3	218)	1 3 6 9 6 6 3
224)	1 3 6 9 7 7 3	231)	1 3 6 9 8 8 3	233)	1 3 6 9 9 3 3
240)	1 3 6 9 10 3 3	247)	1 3 6 9 11 3 3	254)	1 3 6 9 12 3 3
261)	1 3 6 10 3 3 3	263)	1 3 6 10 4 4 3	266)	1 3 6 10 5 5 3
271)	1 3 6 10 6 6 3	277)	1 3 6 10 7 7 3	284)	1 3 6 10 8 8 3
292)	1 3 6 10 9 9 3	293)	1 3 6 10 10 3 3	300)	1 3 6 10 11 3 3
307)	1 3 6 10 12 3 3	314)	1 3 6 10 13 3 3	321)	1 3 6 10 14 3 3
328)	1 3 6 10 15 3 3				

TABLE 2

Theorem 2.10 (Theorem 2.17, [9]). *Let h_{d-2} , h_{d-1} , h_d be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d-1 \rangle} \quad \text{and} \quad h_{d-1} = h_{d-2}^{\langle d-2 \rangle} .$$

Let I be any ideal in $R = k[x_1, \dots, x_n]$ such that the Hilbert function of R/I satisfies

$$\begin{aligned} \mathbf{H}(R/I, d-2) &= h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) &= h_{d-1}, \\ \mathbf{H}(R/I, d) &= h_d . \end{aligned}$$

Then, the ring R/I has socle of dimension ε in degree $d-2$.

Using Theorem 2.10, one can show that the following 22 cases in Table 3 are not level O -sequences.

4)	1 3 3 4 4 4 3	6)	1 3 3 4 5 4 3	7)	1 3 3 4 5 5 3
8)	1 3 3 4 5 6 3	14)	1 3 4 4 5 4 3	15)	1 3 4 4 5 5 3
16)	1 3 4 4 5 6 3	18)	1 3 4 5 4 3 3	23)	1 3 4 5 5 6 3
33)	1 3 5 4 5 3 3	34)	1 3 5 4 5 4 3	35)	1 3 5 4 5 5 3
36)	1 3 5 4 5 6 3	43)	1 3 5 5 5 6 3	45)	1 3 5 5 6 4 3
46)	1 3 5 5 6 5 3	47)	1 3 5 5 6 6 3	48)	1 3 5 5 6 7 3
50)	1 3 5 6 4 3 3	52)	1 3 5 6 5 3 3	55)	1 3 5 6 5 6 3
60)	1 3 5 6 6 7 3	68)	1 3 5 7 4 3 3	70)	1 3 5 7 5 3 3
73)	1 3 5 7 5 6 3	74)	1 3 5 7 6 3 3	84)	1 3 5 7 7 8 3
100)	1 3 6 4 3 3 3	103)	1 3 6 4 5 3 3	104)	1 3 6 4 5 4 3
105)	1 3 6 4 5 5 3	106)	1 3 6 4 5 6 3	107)	1 3 6 5 3 3 3
108)	1 3 6 5 4 3 3	114)	1 3 6 5 6 3 3	115)	1 3 6 5 6 4 3
116)	1 3 6 5 6 5 3	117)	1 3 6 5 6 6 3	118)	1 3 6 5 6 7 3
125)	1 3 6 6 5 6 3	130)	1 3 6 6 6 7 3	138)	1 3 6 7 4 3 3
140)	1 3 6 7 5 3 3	143)	1 3 6 7 5 6 3	144)	1 3 6 7 6 3 3
148)	1 3 6 7 6 7 3	154)	1 3 6 7 7 8 3	170)	1 3 6 8 4 3 3
172)	1 3 6 8 5 3 3	175)	1 3 6 8 5 6 3	176)	1 3 6 8 6 3 3
180)	1 3 6 8 6 7 3	181)	1 3 6 8 7 3 3	186)	1 3 6 8 7 8 3
209)	1 3 6 9 4 3 3	211)	1 3 6 9 5 3 3	214)	1 3 6 9 5 6 3
215)	1 3 6 9 6 3 3	219)	1 3 6 9 6 7 3	220)	1 3 6 9 7 3 3
223)	1 3 6 9 7 6 3	225)	1 3 6 9 7 8 3	226)	1 3 6 9 8 3 3
262)	1 3 6 10 4 3 3	264)	1 3 6 10 5 3 3	267)	1 3 6 10 5 6 3
268)	1 3 6 10 6 3 3	273)	1 3 6 10 7 3 3	279)	1 3 6 10 8 3 3
286)	1 3 6 10 9 3 3				

TABLE 3

Proposition 2.11 (Proposition 2.11, [1]). *The Artinian O -sequence*

$\mathbf{H} : 1 \ 3 \ 6 \ 7 \ 5 \ 4 \ 3$ *is not level.*

Proposition 2.12 (Proposition 2.12, [1]). *The Artinian O-sequence*

$\mathbf{H} : 1 \ 3 \ 6 \ 8 \ 6 \ 5 \ 3$ *is not level.*

By the similar argument as above, we can show that the following cases in Table 4 are not level.

22)	1 3 4 5 5 5 3	71)	1 3 5 7 5 4 3	141)	1 3 6 7 5 4 3
178)	1 3 6 8 6 5 3	269)	1 3 6 10 6 4 3	276)	1 3 6 10 7 6 3

TABLE 4

Proposition 2.13 (Proposition 3.9, [10]). *Let h_{d-2}, h_{d-1}, h_d be three integers such that*

$$h_d = h_{d-1}^{\langle d-1 \rangle} \quad \text{and} \quad h_{d-1} = h_{d-2}^{\langle d-2 \rangle} .$$

Let I be any ideal in $R = k[x_1, \dots, x_n]$ for which

$$\begin{aligned} \mathbf{H}(R/I, d-2) &= h_{d-2} + \varepsilon, \quad \varepsilon \geq n, \\ \mathbf{H}(R/I, d-1) &= h_{d-1}, \\ \mathbf{H}(R/I, d) &= h_d - 1. \end{aligned}$$

Then $\dim_k \text{soc}(R/I)_{d-2} \geq 1$ and so any O-sequence

$$(1, n, \dots, h_{d-2} + \varepsilon, h_{d-1}, h_d - 1, \dots, h_s)$$

of length $s+1$ ($s \geq d$) is not a level O-sequence.

The following cases in Table 5 are not level by Proposition 2.13.

80)	1 3 5 7 7 4 3	132)	1 3 6 6 7 4 3	150)	1 3 6 7 7 4 3
156)	1 3 6 7 8 4 3	163)	1 3 6 7 9 4 3	182)	1 3 6 8 7 4 3
188)	1 3 6 8 8 4 3	195)	1 3 6 8 9 4 3	221)	1 3 6 9 7 4 3
227)	1 3 6 9 8 4 3	234)	1 3 6 9 9 4 3	241)	1 3 6 9 10 4 3
248)	1 3 6 9 11 4 3	274)	1 3 6 10 7 4 3	280)	1 3 6 10 8 4 3
287)	1 3 6 10 9 4 3	294)	1 3 6 10 10 4 3	301)	1 3 6 10 11 4 3

TABLE 5

Proposition 2.14 (Proposition 2.14, [1]). *The Artinian O -sequence $\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 8 \ 7 \ 3$ is not level.*

Question 2.15. Let J be a lex-segment ideal of R such that the Hilbert function of R/J is $\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 8 \ 7 \ 3$. Then the minimal free resolution of R/J is

$$\begin{aligned}
0 &\rightarrow \mathbf{R}^3(-6) \oplus R^2(-7) \oplus R^4(-8) \oplus R^3(-9) \\
&\rightarrow R^9(-5) \oplus \mathbf{R}^4(-6) \oplus R^9(-7) \oplus R^6(-8) \\
&\rightarrow R^7(-4) \oplus R^2(-5) \oplus R^5(-6) \oplus R^3(-7) \\
&\rightarrow R \rightarrow R/I \rightarrow 0.
\end{aligned}$$

Note that the difference between the Betti number of the shift 6 in the last free module and the Betti number of the same shift in the middle free module is 1. However, Ahn proved [1] that \mathbf{H} cannot be level as above. We have a few more cases similar to \mathbf{H} in Proposition 2.14. For example, if I is the lex-segment ideal of R with Hilbert function

$\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 7 \ 5 \ 3$, then R/I has a minimal free resolution as follows.

$$\begin{aligned}
0 &\rightarrow \mathbf{R}^4(-6) \oplus R^2(-7) \oplus R^2(-8) \oplus R^3(-9) \\
&\rightarrow R^{11}(-5) \oplus \mathbf{R}^5(-6) \oplus R^4(-7) \oplus R^6(-8) \\
&\rightarrow R^8(-4) \oplus R^3(-5) \oplus R^2(-6) \oplus R^3(-7) \\
&\rightarrow R \rightarrow R/I \rightarrow 0.
\end{aligned}$$

53)	1 3 5 6 5 4 3	76)	1 3 5 7 6 5 3	123)	1 3 6 6 5 4 3
146)	1 3 6 7 6 5 3	216)	1 3 6 9 6 4 3	275)	1 3 6 10 7 5 3

TABLE 6

We now prove that some of them in Table 6 cannot be level using a different idea from those of Proposition 2.14.

Recall the Betti diagram of R/I :

$$\begin{array}{cccc}
& & 0 & 1 & 2 \\
0 & \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & * & * & * \\
\vdots & \vdots & \vdots & \vdots \\
d-3 & 0 & \beta_{0,d-2}(I) & \beta_{1,d-1}(I) & \beta_{2,d}(I) \\
d-2 & 0 & \beta_{0,d-1}(I) & \beta_{1,d}(I) & \beta_{2,d+1}(I) \\
d-1 & 0 & \beta_{0,d}(I) & \beta_{1,d+1}(I) & \beta_{2,d+2}(I) \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array} \right)
\end{array}$$

In other words, we shall show that if R/I is a codimension 3 Artinian algebra of type 3 and socle degree 6 with Hilbert function \mathbf{H} such that

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) + 1, \quad \text{for } d = 4, 5$$

where I^{lex} is a lex-segment ideal associated to I (see Propositions 2.16 and 2.18), then a couple of cases in Table 6 cannot be level.

Let $R = k[x_1, x_2, x_3]$. Consider a codimension 3 graded Artinian algebra R/I with Hilbert function $\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 8 \ 7 \ 3$ satisfying the condition $\beta_{1,6}(I^{\text{lex}}) = \beta_{2,6}(I^{\text{lex}}) + 1$. First of all, we will prove that R/I cannot be level, in the following Proposition.

Proposition 2.16. *The Artinian O-sequence $\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 8 \ 7 \ 3$ cannot be level.*

Before we give the proof of this proposition, we recall the following theorem here.

Theorem 2.17 (Eliahou-Kervaire, [2]). *Let I be a stable monomial ideal of R . Denote by $\mathcal{G}(I)$ the set of minimal (monomial) generators of I and $\mathcal{G}(I)_d$ the elements of $\mathcal{G}(I)$ having degree d . Then*

$$\beta_{q,i}(I) = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T) - 1}{q}$$

Proof of Proposition 2.16. We assume that there exists an ideal I in R such that R/I is an Artinian level algebra with the given Hilbert function \mathbf{H} .

Let I^{lex} be a lex-segment ideal associated to I in R . Then the Betti diagram of R/I^{lex} is as follows:

$$\begin{array}{cccc}
 \text{total:} & 1 & - & - & - \\
 \hline
 0: & 1 & - & - & - \\
 1: & 0 & 0 & 0 & 0 \\
 2: & 0 & 0 & 0 & 0 \\
 3: & 0 & 7 & 9 & (3) \\
 4: & 0 & 2 & (4) & 2 \\
 5: & 0 & 5 & 9 & 4 \\
 6: & 0 & 3 & 6 & 3
 \end{array}$$

In other words, we have

$$\beta_{1,6}(I^{\text{lex}}) = \beta_{2,6}(I^{\text{lex}}) + 1 = 4.$$

Let $K := (I_{\leq 5})^{\text{lex}}$. Note that I^{lex} and K agree in degree ≤ 5 . Hence we can write the Betti diagram of R/K as

total:	1	-	-	-

0:	1	-	-	-
1:	0	0	0	0
2:	0	0	0	0
3:	0	7	9	(3)
4:	0	2	(4)	2
5:	0	x	y	*
			

Since R/I is level and $(I_{\leq 5})$ has no generators in degree 6, we have

$$\beta_{0,6}((I_{\leq 5})) = \beta_{2,6}((I_{\leq 5})) = 0.$$

Moreover, by Lemma 2.9 (a) in [1], we see that

$$\begin{aligned} x &= \beta_{0,6}(K) = \beta_{1,6}(K) - \beta_{1,6}(I_{\leq 5}) - \beta_{2,6}(K) \\ &\leq \beta_{1,6}(K) - \beta_{2,6}(K) = 4 - 3 = 1, \end{aligned}$$

which follows that $x = 0$ or $x = 1$.

Case 1. Let $x = 0$. Then, by Theorem 2.17, we have $y = 0$. Moreover, by Lemma 2.9 (a) in [1] again, we have

$$\beta_{2,7}(K) - \beta_{2,7}((I_{\leq 5})) \leq \beta_{1,7}(K) - \beta_{1,7}((I_{\leq 5})) \leq \beta_{1,7}(K) = y = 0,$$

which follows that $\beta_{2,7}(K) = \beta_{2,7}((I_{\leq 5})) = 2$. Hence $R/(I_{\leq 5})$ has two dimensional socle elements in degree 4, so does R/I since $R/(I_{\leq 5})$ and R/I agree in degree ≤ 5 , which is a contradiction.

Case 2. Let $x = 1$. Then K has one generator in degree 6, and hence the Hilbert functions of R/K and $R/(I_{\leq 5})$ begin

$$\mathbf{G} : 1 \ 3 \ 6 \ 10 \ 8 \ 7 \ 7 \ \dots$$

It follows from Theorem 4.1 in [1] that any graded Artinian algebra with Hilbert function \mathbf{G} has at least one socle element in degree 4, and hence so does R/I since R/I and $R/(I_{\leq 5})$ agree in degree ≤ 5 , which is a contradiction.

By Cases 1 and 2, the Hilbert function \mathbf{H} cannot be level, as we desired. □

Now we prove one more case in Table 6 which we can use the similar idea as in the proof of Proposition 2.16.

Proposition 2.18. *The Artinian O-sequence $\mathbf{H} : 1 \ 3 \ 6 \ 9 \ 6 \ 4 \ 3$ cannot be level.*

Proof. We assume that there exists an ideal I in $R = k[x_1, x_2, x_3]$ such that R/I is an Artinian level algebra with the given Hilbert function \mathbf{H} .

Let I^{lex} be a lex-segment ideal in $R = k[x_1, x_2, x_3]$ associated to I . Then the Betti diagram of R/I^{lex} is as follows:

total:	1	-	-	-
0:	1	-	-	-
1:	0	0	0	0
2:	0	1	0	0
3:	0	6	10	4
4:	0	5	5	(2)
5:	0	1	(2)	1
6:	0	3	6	3

In other words, we have

$$\beta_{1,7}(I^{\text{lex}}) = \beta_{2,7}(I^{\text{lex}}) = 2.$$

Let $K := (I_{\leq 6})^{\text{lex}}$. Note that I^{lex} and K agree in degree ≤ 6 . Hence we can write the Betti diagram of R/K as

total:	1	-	-	-
0:	1	-	-	-
1:	0	0	0	0
2:	0	1	0	0
3:	0	6	10	4
4:	0	5	5	(2)
5:	0	1	(2)	1
6:	0	x	y	*

Since R/I is level and $(I_{\leq 6})$ has no generators in degree 7, we have

$$\beta_{0,7}((I_{\leq 6})) = \beta_{2,7}((I_{\leq 6})) = 0.$$

Moreover, by Lemma 2.9 (a) in [1], we see that

$$\begin{aligned} x &= \beta_{0,7}(K) \\ &= \beta_{1,7}(K) - \beta_{1,7}(I_{\leq 6}) - \beta_{2,7}(K) \\ &\leq \beta_{1,7}(K) - \beta_{2,7}(K) \\ &= 2 - 2 \\ &= 0, \end{aligned}$$

which follows that $x = 0$. Then, by Theorem 2.17, we have $y = 0$.

Moreover, by Lemma 2.9 (a) in [1] again, we have

$$\begin{aligned} \beta_{2,8}(K) - \beta_{2,8}((I_{\leq 6})) &\leq \beta_{1,8}(K) - \beta_{1,8}((I_{\leq 6})) \\ &\leq \beta_{1,8}(K) \\ &= y \\ &= 0, \end{aligned}$$

which follows that

$$\beta_{2,8}(K) = \beta_{2,8}((I_{\leq 6})) = 1.$$

Hence $R/(I_{\leq 6})$ has one dimensional socle elements in degree 5, so does R/I since $R/(I_{\leq 6})$ and R/I agree in degree ≤ 6 , which is a contradiction.

Therefore, R/I is not level, as we wished. □

Remark 2.19. Consider an O-sequence $\mathbf{H} : 1 \ 3 \ 5 \ 6 \ 6 \ 4 \ 3$. Let I be a lex-segment ideal of $R = k[x_1, x_2, x_3]$ such that the Hilbert function of R/I is H . Then the minimal free resolution of R/I is

$$\begin{aligned}
0 &\rightarrow R(-6) \oplus R^2(-7) \oplus R(-8) \oplus R^3(-9) \\
&\rightarrow R(-4) \oplus R^5(-6) \oplus R^2(-7) \oplus R^6(-8) \\
&\rightarrow R(-2) \oplus R(-3) \oplus R^3(-5) \oplus R(-6) \oplus R^3(-7) \\
&\rightarrow R \rightarrow R/I \rightarrow 0.
\end{aligned}$$

Then we have $\beta_{2,7}(I) = \beta_{1,7}(I) = 2$. Hence we can show that H cannot not level using the same ideal as the proof of Proposition 2.18. In other words, the following cases are the same cases as above, which cannot be level(see Table 7).

57)	1 3 5 6 6 4 3	94)	1 3 5 7 9 5 3	127)	1 3 6 6 6 4 3
145)	1 3 6 7 6 4 3	196)	1 3 6 8 9 5 3	235)	1 3 6 9 9 5 3
257)	1 3 6 9 12 6 3	288)	1 3 6 10 9 5 3	310)	1 3 6 10 12 6 3
322)	1 3 6 10 15 7 3				

TABLE 7

However, the following 68 cases are still open which we should prove.

21)	1 3 4 5 5 4 3	25)	1 3 4 5 6 4 3	26)	1 3 4 5 6 5 3
41)	1 3 5 5 5 4 3	50)	1 3 5 6 4 3 3	52)	1 3 5 6 5 3 3
55)	1 3 5 6 5 6 3	58)	1 3 5 6 6 5 3	63)	1 3 5 6 7 5 3
66)	1 3 5 6 7 8 3	80)	1 3 5 7 7 4 3	81)	1 3 5 7 7 5 3
87)	1 3 5 7 8 5 3	88)	1 3 5 7 8 6 3	89)	1 3 5 7 8 7 3
91)	1 3 5 7 8 9 3	95)	1 3 5 7 9 6 3	96)	1 3 5 7 9 7 3
97)	1 3 5 7 9 8 3	132)	1 3 6 6 7 4 3	150)	1 3 6 7 7 4 3
156)	1 3 6 7 8 4 3	157)	1 3 6 7 8 5 3	158)	1 3 6 7 8 6 3
161)	1 3 6 7 8 9 3	163)	1 3 6 7 9 4 3	177)	1 3 6 8 6 4 3
182)	1 3 6 8 7 4 3	183)	1 3 6 8 7 5 3	184)	1 3 6 8 7 6 3
188)	1 3 6 8 8 4 3	189)	1 3 6 8 8 5 3	195)	1 3 6 8 9 4 3
197)	1 3 6 8 9 6 3	204)	1 3 6 8 10 6 3	205)	1 3 6 8 10 7 3
206)	1 3 6 8 10 8 3	221)	1 3 6 9 7 4 3	222)	1 3 6 9 7 5 3
227)	1 3 6 9 8 4 3	228)	1 3 6 9 8 5 3	230)	1 3 6 9 8 7 3
234)	1 3 6 9 9 4 3	236)	1 3 6 9 9 6 3	241)	1 3 6 9 10 4 3
243)	1 3 6 9 10 6 3	248)	1 3 6 9 11 4 3	250)	1 3 6 9 11 6 3
251)	1 3 6 9 11 7 3	258)	1 3 6 9 12 7 3	259)	1 3 6 9 12 8 3
274)	1 3 6 10 7 4 3	280)	1 3 6 10 8 4 3	281)	1 3 6 10 8 5 3
282)	1 3 6 10 8 6 3	287)	1 3 6 10 9 4 3	289)	1 3 6 10 9 6 3
291)	1 3 6 10 9 8 3	294)	1 3 6 10 10 4 3	296)	1 3 6 10 10 6 3
301)	1 3 6 10 11 4 3	303)	1 3 6 10 11 6 3	308)	1 3 6 10 12 4 3
311)	1 3 6 10 12 7 3	318)	1 3 6 10 13 7 3	325)	1 3 6 10 14 7 3
326)	1 3 6 10 14 8 3	333)	1 3 6 10 15 8 3		

TABLE 8

3. The Construction of Some Level O-sequences

of Type 3 and Socle degree 7.

In this appendix, we introduce how to construct level O-sequences using the “link-sum” construction which are given by Cho in [16] and Yoo in [19]. They used the computer programs S-plus and CoCoA to produce them here.

Definition 3.1. (Definition 2.1, [18])

- (a) A finite set \mathbb{X} of points \mathbb{P}^2 is called a **basic configuration** of type (d, e) if there exists distinct elements b_j, c_j in k such that

$$I_{\mathbb{X}} = \left(\prod_{j=1}^d (x - b_j z), \prod_{j=1}^e (y - c_j z) \right).$$

We denote $\mathbb{X} := \mathbb{B}(d, e)$.

- (b) A finite set \mathbb{X} of points in \mathbb{P}^2 is called a **pure configuration** if there exist finite basic configurations $\mathbb{B}(d_1, e_1), \dots, \mathbb{B}(d_m, e_m)$ where $e_1 > \dots > e_m$, which satisfy the following three conditions:

- i) $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \emptyset$ if $i \neq j$,
- ii) $\mathbb{X} = \mathbb{B}(d_1, e_1) \cup \dots \cup \mathbb{B}(d_m, e_m)$,
- iii) $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$ for all $1 \leq i \leq m - 1$, where $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \rightarrow \mathbb{P}^1$ is the map defined by sending the point (x, y, z) to the point (y, z) . In this case, we denote $\mathbb{X} = \cup_{i=1}^m \mathbb{B}(d_i, e_i)$.

Proposition 3.2 (Proposition 3.8 in [17]). *Let $\mathbb{X} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ be a pure configuration in \mathbb{P}^2 . Then a minimal free resolution of \mathbb{X} is :*

$$0 \rightarrow \bigoplus_{i=1}^m R(-p_i) \rightarrow \bigoplus_{i=1}^{m+1} R(-q_i) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0,$$

where

$$q_1 = e_1, \quad q_i = d_1 + \cdots + d_{i-1} + e_i \quad (2 \leq i \leq m),$$

$$q_{m+1} = d_1 + \cdots + d_m, \quad p_i = q_i + d_i \quad (1 \leq i \leq m).$$

Example 3.3 (Example 3.9 in [17]). Let $\mathbb{X} = \mathbb{B}(3, 6) \cup \mathbb{B}(4, 2) \cup \mathbb{B}(2, 2) \cup \mathbb{B}(1, 1)$ be a pure configuration in \mathbb{P}^2 .

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \right\}$$

The minimal free resolution of \mathbb{Z} is

$$0 \rightarrow R^4(-9) \rightarrow R(-6) \oplus R^2(-7) \oplus R^2(-8) \rightarrow R \rightarrow R/I \rightarrow 0.$$

Corollary 3.4 (Corollary 3.10, [17]). *Let $\mathbb{X} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ be a pure configuration in \mathbb{P}^2 . Then \mathbb{X} is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all $1 \leq i \leq m - 1$.

If $r(A)$ is the Cohen-Macaulay type of a Cohen-Macaulay standard graded k -algebra A , and if $A = \bigoplus_{i \geq 0} A_i$ is an Artinian level algebra, then $r(A) = \dim_k A_{\sigma(A)-1}$, where $\sigma(A) = \min\{i | A_i = 0\}$. If $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ is a level pure configuration in \mathbb{P}^2 , then $r(\mathbb{Z}) = r(R/I_{\mathbb{Z}}) = m$.

Lemma 3.5 (Lemma 3.14, [17]). *Let \mathbb{Z} be a level set of points in \mathbb{P}^n and \mathbb{X} a subset of \mathbb{Z} . Set $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is an Artinian level graded k -algebra with $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \sigma(\mathbb{Z}) - 1$ and $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq r(\mathbb{Z})$.*

Corollary 3.6 (Corollary 3.15, [17]). *Let $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ be a level pure configuration in \mathbb{P}^2 and \mathbb{X} a subset of \mathbb{Z} . Set $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is an Artinian level graded k -algebra with $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = d_1 + e_1 - 2$ and $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq m$.*

Let \mathbb{X} be the set of all \bullet 's in \mathbb{Z} and \mathbb{Y} be the set of all $*$'s in \mathbb{Z} . Next to each diagram we give the Hilbert functions of \mathbb{X} , \mathbb{Y} , \mathbb{Z} and $A = R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$.

42) [1,3,5,5,5,5,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ \bullet & \bullet & * & & & & & \\ * & * & * & * & * & & & \\ * & * & \bullet & * & * & & & \\ * & * & * & * & * & \bullet & * & \\ * & * & * & \bullet & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 5 \ 5 \ 5 \ 5 \ 5 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 25 \ 25 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 5 \ 5 \ 5 \ 3 \ 0 \rightarrow . \end{array}$$

59) [1,3,5,6,6,6,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & * & & & & & \\ * & * & * & & & & & \\ * & * & * & \bullet & * & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & \bullet & * & \\ * & \bullet & \bullet & \bullet & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 6 \ 6 \ 6 \ 6 \ 6 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ 24 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 6 \ 6 \ 6 \ 3 \ 0 \rightarrow . \end{array}$$

64) [1,3,5,6,7,6,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ * & \bullet & * & * & * & & & \\ * & * & * & * & * & * & * & \\ * & * & * & * & * & * & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 6 \ 7 \ 7 \ 7 \ 7 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 23 \ 23 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 6 \ 7 \ 6 \ 3 \ 0 \rightarrow . \end{array}$$

65) [1,3,5,6,7,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & \bullet & & & & & \\ * & \bullet & * & * & * & & & \\ * & * & * & * & * & & & \\ \bullet & * & \bullet & \bullet & \bullet & * & \bullet & \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 7 \ 7 \ 7 \ 7 \ 7 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 23 \ 23 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 6 \ 7 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

82) [1,3,5,7,7,6,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & * & * & & & & & \\ * & \bullet & * & & & & & \\ \bullet & \bullet & \bullet & \bullet & * & & & \\ * & * & * & * & * & & & \\ * & * & * & * & \bullet & * & * & \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 7 \ 7 \ 7 \ 7 \ 7 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 23 \ 23 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 7 \ 7 \ 6 \ 3 \ 0 \rightarrow . \end{array}$$

83) [1,3,5,7,7,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & \bullet & & & & & \\ * & * & * & & & & & \\ * & * & * & \bullet & * & & & \\ * & * & \bullet & \bullet & * & & * & * \\ * & * & \bullet & \bullet & * & * & * & * \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 7 \ 7 \ 7 \ 7 \ 7 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 23 \ 23 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 7 \ 7 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

90) [1,3,5,7,8,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & \bullet & & & & & \\ * & * & \bullet & & & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & & & \\ \bullet & * & \bullet & \bullet & * & \bullet & \bullet & \\ * & * & \bullet & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ 8 \ 8 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 22 \ 22 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

98) [1,3,5,7,9,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & & & \\ \bullet & \bullet & \bullet & * & * & \bullet & * & \\ \bullet & \bullet & \bullet & \bullet & * & * & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 5 \ 7 \ 9 \ 9 \ 9 \ 9 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 21 \ 21 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 5 \ 7 \ 9 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

128) [1,3,6,6,6,5,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & * & * & * & \bullet & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & * & * & * & * & \\ * & * & \bullet & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 6 \ 6 \ 6 \ 6 \ 6 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 24 \ 24 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 6 \ 6 \ 5 \ 3 \ 0 \rightarrow . \end{array}$$

129) [1,3,6,6,6,6,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & \bullet & * & & & \\ \bullet & * & * & * & * & & & \\ * & * & \bullet & \bullet & * & * & * & \\ * & * & * & * & * & \bullet & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 6 \ 6 \ 6 \ 6 \ 6 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ 24 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 6 \ 6 \ 6 \ 3 \ 0 \rightarrow . \end{array}$$

151) [1,3,6,7,7,5,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & * & \bullet & & & & & \\ * & * & * & * & * & & & \\ * & \bullet & * & * & * & & & \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & \bullet & * & * \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 7 \ 7 \ 7 \ 7 \ 7 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 23 \ 23 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 7 \ 7 \ 5 \ 3 \ 0 \rightarrow . \end{array}$$

152) [1,3,6,7,7,6,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ * & * & * & & & & & \\ * & * & * & * & \bullet & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & \bullet & \bullet & \\ * & * & * & * & * & * & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 7 \ 7 \ 7 \ 7 \ 7 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 23 \ 23 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 7 \ 7 \ 6 \ 3 \ 0 \rightarrow . \end{array}$$

153) [1,3,6,7,7,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & \bullet & & & & & \\ \bullet & * & \bullet & & & & & \\ \bullet & \bullet & * & * & * & & & \\ * & * & * & * & \bullet & & & \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & \bullet & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 7 \ 7 \ 7 \ 7 \ 7 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 23 \ 23 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 7 \ 7 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

159) [1,3,6,7,8,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & * & * & & & & & \\ \bullet & * & * & & & & & \\ \bullet & * & \bullet & * & * & & & \\ \bullet & \bullet & * & * & * & & & \\ * & * & * & \bullet & * & * & * & * \\ \bullet & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 7 \ 8 \ 8 \ 8 \ 8 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 22 \ 22 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 7 \ 8 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

160) [1,3,6,7,8,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & \bullet & * & & & \\ * & * & * & * & * & & & \\ * & \bullet & \bullet & \bullet & \bullet & \bullet & * & * \\ * & \bullet & \bullet & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 7 \ 8 \ 8 \ 8 \ 8 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 22 \ 22 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 7 \ 8 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

190) [1,3,6,8,8,6,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccc} * & \bullet & \bullet & & & & & & \\ \bullet & \bullet & * & & & & & & \\ * & * & * & * & * & & & & \\ * & \bullet & \bullet & \bullet & * & & & & \\ * & * & * & * & \bullet & * & * & & \\ * & * & * & * & * & * & * & & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 8 \ 8 \ 8 \ 8 \ 8 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 22 \ 22 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 8 \ 8 \ 6 \ 3 \ 0 \rightarrow . \end{array}$$

191) [1,3,6,8,8,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccc} \bullet & \bullet & \bullet & & & & & & \\ * & * & * & & & & & & \\ * & * & * & * & * & & & & \\ * & * & * & \bullet & \bullet & & & & \\ * & * & * & * & * & * & * & & \\ \bullet & * & * & * & \bullet & \bullet & * & & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 8 \ 8 \ 8 \ 8 \ 8 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 22 \ 22 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 8 \ 8 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

192) [1,3,6,8,8,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccc} * & \bullet & \bullet & & & & & & \\ \bullet & * & * & & & & & & \\ * & * & * & * & * & & & & \\ * & * & * & \bullet & * & & & & \\ * & \bullet & * & * & * & \bullet & \bullet & & \\ * & * & * & * & * & * & \bullet & & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 8 \ 8 \ 8 \ 8 \ 8 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 22 \ 22 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 8 \ 8 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

198) [1,3,6,8,9,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccc} \bullet & \bullet & \bullet & & & & & & \\ \bullet & \bullet & * & & & & & & \\ \bullet & \bullet & * & * & * & & & & \\ \bullet & * & * & * & * & & & & \\ * & * & * & * & * & * & * & & \\ \bullet & * & * & * & * & * & * & & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 8 \ 9 \ 9 \ 9 \ 9 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 21 \ 21 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 8 \ 9 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

199) [1,3,6,8,9,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccc} * & * & \bullet & & & & & & \\ * & * & * & & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\ \bullet & * & * & * & * & & & & \\ \bullet & * & * & * & \bullet & * & * & & \\ * & * & * & * & * & * & * & & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 8 \ 9 \ 9 \ 9 \ 9 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 21 \ 21 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 8 \ 9 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

200) [1,3,6,8,9,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & \bullet & & & & & \\ \bullet & * & \bullet & & & & & \\ \bullet & * & \bullet & * & * & & & \\ * & * & \bullet & * & * & & & \\ * & * & \bullet & * & \bullet & * & * & \\ * & * & \bullet & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 8 \ 9 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 8 \ 9 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

207) [1,3,6,8,10,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & * & * & & & & & \\ * & * & * & & & & & \\ \bullet & * & * & * & * & & & \\ \bullet & * & \bullet & * & * & & & \\ \bullet & \bullet & * & \bullet & \bullet & * & \bullet & \\ \bullet & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 8 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 8 \ 10 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

229) [1,3,6,9,8,6,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & \bullet & * & * & \bullet & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & \bullet & * & \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ 9 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 21 \ 21 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 8 \ 6 \ 3 \ 0 \rightarrow . \end{array}$$

237) [1,3,6,9,9,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & * & \bullet & & & & & \\ \bullet & * & * & & & & & \\ * & * & \bullet & \bullet & \bullet & & & \\ * & \bullet & * & \bullet & * & & & \\ * & * & * & * & * & * & * & \\ \bullet & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ 9 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 21 \ 21 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 9 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

238) [1,3,6,9,9,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & \bullet & * & * & \bullet & & & \\ * & * & * & \bullet & \bullet & & & \\ * & * & * & * & \bullet & * & \bullet & \\ * & * & \bullet & \bullet & * & * & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ 9 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 21 \ 21 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 9 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

239) [1,3,6,9,9,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & * & & & & & \\ * & \bullet & * & & & & & \\ * & * & * & * & * & & & \\ * & * & \bullet & \bullet & \bullet & & & \\ * & * & * & \bullet & \bullet & * & * & \\ * & * & * & * & \bullet & * & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ 9 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 21 \ 21 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

244) [1,3,6,9,10,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & * & & & & & \\ * & * & \bullet & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ * & * & * & \bullet & \bullet & & & \\ * & * & \bullet & * & * & * & * & \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 18 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 10 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

245) [1,3,6,9,10,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & * & & & & & \\ * & * & \bullet & & & & & \\ \bullet & \bullet & * & \bullet & \bullet & & & \\ * & * & * & * & * & & & \\ * & * & * & * & * & * & * & \\ \bullet & \bullet & \bullet & \bullet & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 10 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

246) [1, 3, 6, 9, 10, 9, 3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & \bullet & & & & & \\ \bullet & * & \bullet & * & * & & & \\ * & \bullet & \bullet & \bullet & * & & & \\ * & * & \bullet & * & * & * & \bullet & \\ * & \bullet & \bullet & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 10 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

252) [1,3,6,9,11,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & * & & & & & \\ * & * & * & & & & & \\ * & * & * & * & * & & & \\ \bullet & * & * & * & * & & & \\ \bullet & * & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & * \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 11 \ 13 \ 14 \ 14 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 16 \ 16 \ 16 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 11 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

253) [1,3,6,9,11,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & * & & & & & \\ \bullet & \bullet & * & & & & & \\ \bullet & \bullet & \bullet & * & * & & & \\ * & \bullet & * & * & \bullet & & & \\ * & * & * & * & * & * & \bullet & \\ * & \bullet & * & * & * & \bullet & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 11 \ 11 \ 11 \ 11 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 19 \ 19 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 11 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

260) [1,3,6,9,12,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ \bullet & \bullet & * & & & & & \\ \bullet & * & * & \bullet & * & & & \\ \bullet & * & * & * & \bullet & & & \\ \bullet & * & * & * & * & \bullet & * & \\ \bullet & \bullet & * & \bullet & \bullet & * & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 9 \ 12 \ 13 \ 13 \ 13 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 17 \ 17 \ 17 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 9 \ 12 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

290) [1,3,6,10,9,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ \bullet & * & * & * & * & & & \\ \bullet & * & \bullet & * & * & & & \\ * & * & * & \bullet & * & * & * & \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 9 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

297) [1,3,6,10,10,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & \bullet & \bullet & & & & & \\ * & \bullet & \bullet & & & & & \\ \bullet & * & \bullet & * & \bullet & & & \\ \bullet & \bullet & * & * & * & & & \\ * & * & * & * & * & * & \bullet & \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 18 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 10 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

298) [1,3,6,10,10,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} & * & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & \bullet & * & \bullet & * & & & \\ \bullet & * & * & * & * & & & \\ * & * & * & * & * & \bullet & \bullet & \\ * & * & \bullet & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 10 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

299) [1,3,6,10,10,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & & & & \\ * & \bullet & * & \bullet & * & & & \\ \bullet & * & \bullet & \bullet & \bullet & & & \\ * & * & * & * & \bullet & * & * & \\ * & \bullet & * & * & * & \bullet & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ 10 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 20 \ 20 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 10 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

304) [1,3,6,10,11,7,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ * & \bullet & \bullet & & & & & \\ \bullet & * & * & \bullet & \bullet & & & \\ \bullet & \bullet & * & \bullet & \bullet & & & \\ * & * & * & * & * & * & * & \\ * & \bullet & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 13 \ 13 \ 13 \ 13 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 13 \ 15 \ 17 \ 17 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 11 \ 7 \ 3 \ 0 \rightarrow . \end{array}$$

305) [1,3,6,10,11,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & \bullet & * & & & \\ * & \bullet & \bullet & * & * & & & \\ \bullet & \bullet & \bullet & * & * & * & * & \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 13 \ 15 \ 15 \ 15 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 13 \ 14 \ 15 \ 15 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 11 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

306) [1,3,6,10,11,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ \bullet & \bullet & * & & & & & \\ * & \bullet & \bullet & * & * & & & \\ * & * & \bullet & * & * & & & \\ * & \bullet & \bullet & * & \bullet & * & \bullet & \\ \bullet & * & \bullet & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 11 \ 11 \ 11 \ 11 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 19 \ 19 \ 19 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 11 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

312) [1,3,6,10,12,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & * & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ \bullet & * & \bullet & \bullet & \bullet & & & \\ * & \bullet & * & \bullet & \bullet & & & \\ * & * & \bullet & * & * & * & * & \\ * & \bullet & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 14 \ 14 \ 14 \ 14 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 13 \ 15 \ 16 \ 16 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 12 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

313) [1,3,6,10,12,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & \bullet & & & & & \\ * & \bullet & * & & & & & \\ * & \bullet & * & * & \bullet & & & \\ * & \bullet & * & \bullet & * & & & \\ * & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & * & * & * & \bullet & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 13 \ 14 \ 14 \ 14 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 14 \ 16 \ 16 \ 16 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 12 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

319) [1,3,6,10,13,8,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ * & \bullet & \bullet & & & & & \\ \bullet & * & * & * & * & & & \\ \bullet & * & * & \bullet & \bullet & & & \\ \bullet & * & \bullet & \bullet & \bullet & \bullet & \bullet & * \\ * & * & * & * & * & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 14 \ 14 \ 14 \ 14 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 14 \ 15 \ 16 \ 16 \ 16 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 13 \ 8 \ 3 \ 0 \rightarrow . \end{array}$$

320) [1,3,6,10,13,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ * & * & \bullet & * & * & & & \\ \bullet & * & \bullet & * & * & & & \\ * & * & \bullet & * & * & * & \bullet & \bullet \\ \bullet & \bullet & \bullet & * & * & \bullet & \bullet & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 13 \ 14 \ 14 \ 14 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 16 \ 16 \ 16 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 13 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

327) [1,3,6,10,14,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ * & * & * & & & & & \\ * & \bullet & * & * & \bullet & & & \\ * & \bullet & \bullet & \bullet & * & & & \\ * & * & * & * & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & * & * & \bullet & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 14 \ 14 \ 14 \ 14 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 16 \ 16 \ 16 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 14 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

334) [1,3,6,10,15,9,3]

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} * & * & * & & & & & \\ \bullet & * & \bullet & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ * & * & * & * & \bullet & & & \\ \bullet & * & \bullet & * & * & * & \bullet & \bullet \\ \bullet & \bullet & * & \bullet & \bullet & * & * & \end{array} \right\} \begin{array}{l} \mathbf{H}_Z : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 30 \rightarrow \\ \mathbf{H}_X : 1 \ 3 \ 6 \ 10 \ 15 \ 15 \ 15 \ 15 \rightarrow \\ \mathbf{H}_Y : 1 \ 3 \ 6 \ 10 \ 15 \ 15 \ 15 \ 15 \rightarrow \\ \mathbf{H}_A : 1 \ 3 \ 6 \ 10 \ 15 \ 9 \ 3 \ 0 \rightarrow . \end{array}$$

Appendix A

The total 334 Artinian cases based on

Theorem of Fröberg and Laksov in [7].

1)	1 3 3 3 3 3 3	2)	1 3 3 4 3 3 3	3)	1 3 3 4 4 3 3
4)	1 3 3 4 4 4 3	5)	1 3 3 4 5 3 3	6)	1 3 3 4 5 4 3
7)	1 3 3 4 5 5 3	8)	1 3 3 4 5 6 3	9)	1 3 4 3 3 3 3
10)	1 3 4 4 3 3 3	11)	1 3 4 4 4 3 3	12)	1 3 4 4 4 4 3
13)	1 3 4 4 5 3 3	14)	1 3 4 4 5 4 3	15)	1 3 4 4 5 5 3
16)	1 3 4 4 5 6 3	17)	1 3 4 5 3 3 3	18)	1 3 4 5 4 3 3
19)	1 3 4 5 4 4 3	20)	1 3 4 5 5 3 3	21)	1 3 4 5 5 4 3
22)	1 3 4 5 5 5 3	23)	1 3 4 5 5 6 3	24)	1 3 4 5 6 3 3
25)	1 3 4 5 6 4 3	26)	1 3 4 5 6 5 3	27)	1 3 4 5 6 6 3
28)	1 3 4 5 6 7 3	29)	1 3 5 3 3 3 3	30)	1 3 5 4 3 3 3
31)	1 3 5 4 4 3 3	32)	1 3 5 4 4 4 3	33)	1 3 5 4 5 3 3
34)	1 3 5 4 5 4 3	35)	1 3 5 4 5 5 3	36)	1 3 5 4 5 6 3
37)	1 3 5 5 3 3 3	38)	1 3 5 5 4 3 3	39)	1 3 5 5 4 4 3
40)	1 3 5 5 5 3 3	41)	1 3 5 5 5 4 3	42)	1 3 5 5 5 5 3
43)	1 3 5 5 5 6 3	44)	1 3 5 5 6 3 3	45)	1 3 5 5 6 4 3
46)	1 3 5 5 6 5 3	47)	1 3 5 5 6 6 3	48)	1 3 5 5 6 7 3
49)	1 3 5 6 3 3 3	50)	1 3 5 6 4 3 3	51)	1 3 5 6 4 4 3
52)	1 3 5 6 5 3 3	53)	1 3 5 6 5 4 3	54)	1 3 5 6 5 5 3
55)	1 3 5 6 5 6 3	56)	1 3 5 6 6 3 3	57)	1 3 5 6 6 4 3
58)	1 3 5 6 6 5 3	59)	1 3 5 6 6 6 3	60)	1 3 5 6 6 7 3
61)	1 3 5 6 7 3 3	62)	1 3 5 6 7 4 3	63)	1 3 5 6 7 5 3
64)	1 3 5 6 7 6 3	65)	1 3 5 6 7 7 3	66)	1 3 5 6 7 8 3
67)	1 3 5 7 3 3 3	68)	1 3 5 7 4 3 3	69)	1 3 5 7 4 4 3
70)	1 3 5 7 5 3 3	71)	1 3 5 7 5 4 3	72)	1 3 5 7 5 5 3
73)	1 3 5 7 5 6 3	74)	1 3 5 7 6 3 3	75)	1 3 5 7 6 4 3
76)	1 3 5 7 6 5 3	77)	1 3 5 7 6 6 3	78)	1 3 5 7 6 7 3
79)	1 3 5 7 7 3 3	80)	1 3 5 7 7 4 3	81)	1 3 5 7 7 5 3
82)	1 3 5 7 7 6 3	83)	1 3 5 7 7 7 3	84)	1 3 5 7 7 8 3
85)	1 3 5 7 8 3 3	86)	1 3 5 7 8 4 3	87)	1 3 5 7 8 5 3
88)	1 3 5 7 8 6 3	89)	1 3 5 7 8 7 3	90)	1 3 5 7 8 8 3
91)	1 3 5 7 8 9 3	92)	1 3 5 7 9 3 3	93)	1 3 5 7 9 4 3

94)	1 3 5 7 9 5 3	95)	1 3 5 7 9 6 3	96)	1 3 5 7 9 7 3
97)	1 3 5 7 9 8 3	98)	1 3 5 7 9 9 3	99)	1 3 6 3 3 3 3
100)	1 3 6 4 3 3 3	101)	1 3 6 4 4 3 3	102)	1 3 6 4 4 4 3
103)	1 3 6 4 5 3 3	104)	1 3 6 4 5 4 3	105)	1 3 6 4 5 5 3
106)	1 3 6 4 5 6 3	107)	1 3 6 5 3 3 3	108)	1 3 6 5 4 3 3
109)	1 3 6 5 4 4 3	110)	1 3 6 5 5 3 3	111)	1 3 6 5 5 4 3
112)	1 3 6 5 5 5 3	113)	1 3 6 5 5 6 3	114)	1 3 6 5 6 3 3
115)	1 3 6 5 6 4 3	116)	1 3 6 5 6 5 3	117)	1 3 6 5 6 6 3
118)	1 3 6 5 6 7 3	119)	1 3 6 6 3 3 3	120)	1 3 6 6 4 3 3
121)	1 3 6 6 4 4 3	122)	1 3 6 6 5 3 3	123)	1 3 6 6 5 4 3
124)	1 3 6 6 5 5 3	125)	1 3 6 6 5 6 3	126)	1 3 6 6 6 3 3
127)	1 3 6 6 6 4 3	128)	1 3 6 6 6 5 3	129)	1 3 6 6 6 6 3
130)	1 3 6 6 6 7 3	131)	1 3 6 6 7 3 3	132)	1 3 6 6 7 4 3
133)	1 3 6 6 7 5 3	134)	1 3 6 6 7 6 3	135)	1 3 6 6 7 7 3
136)	1 3 6 6 7 8 3	137)	1 3 6 7 3 3 3	138)	1 3 6 7 4 3 3
139)	1 3 6 7 4 4 3	140)	1 3 6 7 5 3 3	141)	1 3 6 7 5 4 3
142)	1 3 6 7 5 5 3	143)	1 3 6 7 5 6 3	144)	1 3 6 7 6 3 3
145)	1 3 6 7 6 4 3	146)	1 3 6 7 6 5 3	147)	1 3 6 7 6 6 3
148)	1 3 6 7 6 7 3	149)	1 3 6 7 7 3 3	150)	1 3 6 7 7 4 3
151)	1 3 6 7 7 5 3	152)	1 3 6 7 7 6 3	153)	1 3 6 7 7 7 3
154)	1 3 6 7 7 8 3	155)	1 3 6 7 8 3 3	156)	1 3 6 7 8 4 3
157)	1 3 6 7 8 5 3	158)	1 3 6 7 8 6 3	159)	1 3 6 7 8 7 3
160)	1 3 6 7 8 8 3	161)	1 3 6 7 8 9 3	162)	1 3 6 7 9 3 3
163)	1 3 6 7 9 4 3	164)	1 3 6 7 9 5 3	165)	1 3 6 7 9 6 3
166)	1 3 6 7 9 7 3	167)	1 3 6 7 9 8 3	168)	1 3 6 7 9 9 3
169)	1 3 6 8 3 3 3	170)	1 3 6 8 4 3 3	171)	1 3 6 8 4 4 3
172)	1 3 6 8 5 3 3	173)	1 3 6 8 5 4 3	174)	1 3 6 8 5 5 3
175)	1 3 6 8 5 6 3	176)	1 3 6 8 6 3 3	177)	1 3 6 8 6 4 3
178)	1 3 6 8 6 5 3	179)	1 3 6 8 6 6 3	180)	1 3 6 8 6 7 3
181)	1 3 6 8 7 3 3	182)	1 3 6 8 7 4 3	183)	1 3 6 8 7 5 3
184)	1 3 6 8 7 6 3	185)	1 3 6 8 7 7 3	186)	1 3 6 8 7 8 3
187)	1 3 6 8 8 3 3	188)	1 3 6 8 8 4 3	189)	1 3 6 8 8 5 3
190)	1 3 6 8 8 6 3	191)	1 3 6 8 8 7 3	192)	1 3 6 8 8 8 3
193)	1 3 6 8 8 9 3	194)	1 3 6 8 9 3 3	195)	1 3 6 8 9 4 3
196)	1 3 6 8 9 5 3	197)	1 3 6 8 9 6 3	198)	1 3 6 8 9 7 3
199)	1 3 6 8 9 8 3	200)	1 3 6 8 9 9 3	201)	1 3 6 8 10 3 3

202)	1 3 6 8 10 4 3	203)	1 3 6 8 10 5 3	204)	1 3 6 8 10 6 3
205)	1 3 6 8 10 7 3	206)	1 3 6 8 10 8 3	207)	1 3 6 8 10 9 3
208)	1 3 6 9 3 3 3	209)	1 3 6 9 4 3 3	210)	1 3 6 9 4 4 3
211)	1 3 6 9 5 3 3	212)	1 3 6 9 5 4 3	213)	1 3 6 9 5 5 3
214)	1 3 6 9 5 6 3	215)	1 3 6 9 6 3 3	216)	1 3 6 9 6 4 3
217)	1 3 6 9 6 5 3	218)	1 3 6 9 6 6 3	219)	1 3 6 9 6 7 3
220)	1 3 6 9 7 3 3	221)	1 3 6 9 7 4 3	222)	1 3 6 9 7 5 3
223)	1 3 6 9 7 6 3	224)	1 3 6 9 7 7 3	225)	1 3 6 9 7 8 3
226)	1 3 6 9 8 3 3	227)	1 3 6 9 8 4 3	228)	1 3 6 9 8 5 3
229)	1 3 6 9 8 6 3	230)	1 3 6 9 8 7 3	231)	1 3 6 9 8 8 3
232)	1 3 6 9 8 9 3	233)	1 3 6 9 9 3 3	234)	1 3 6 9 9 4 3
235)	1 3 6 9 9 5 3	236)	1 3 6 9 9 6 3	237)	1 3 6 9 9 7 3
238)	1 3 6 9 9 8 3	239)	1 3 6 9 9 9 3	240)	1 3 6 9 10 3 3
241)	1 3 6 9 10 4 3	242)	1 3 6 9 10 5 3	243)	1 3 6 9 10 6 3
244)	1 3 6 9 10 7 3	245)	1 3 6 9 10 8 3	246)	1 3 6 9 10 9 3
247)	1 3 6 9 11 3 3	248)	1 3 6 9 11 4 3	249)	1 3 6 9 11 5 3
250)	1 3 6 9 11 6 3	251)	1 3 6 9 11 7 3	252)	1 3 6 9 11 8 3
253)	1 3 6 9 11 9 3	254)	1 3 6 9 12 3 3	255)	1 3 6 9 12 4 3
256)	1 3 6 9 12 5 3	257)	1 3 6 9 12 6 3	258)	1 3 6 9 12 7 3
259)	1 3 6 9 12 8 3	260)	1 3 6 9 12 9 3	261)	1 3 6 10 3 3 3
262)	1 3 6 10 4 3 3	263)	1 3 6 10 4 4 3	264)	1 3 6 10 5 3 3
265)	1 3 6 10 5 4 3	266)	1 3 6 10 5 5 3	267)	1 3 6 10 5 6 3
268)	1 3 6 10 6 3 3	269)	1 3 6 10 6 4 3	270)	1 3 6 10 6 5 3
271)	1 3 6 10 6 6 3	272)	1 3 6 10 6 7 3	273)	1 3 6 10 7 3 3
274)	1 3 6 10 7 4 3	275)	1 3 6 10 7 5 3	276)	1 3 6 10 7 6 3
277)	1 3 6 10 7 7 3	278)	1 3 6 10 7 8 3	279)	1 3 6 10 8 3 3
280)	1 3 6 10 8 4 3	281)	1 3 6 10 8 5 3	282)	1 3 6 10 8 6 3
283)	1 3 6 10 8 7 3	284)	1 3 6 10 8 8 3	285)	1 3 6 10 8 9 3
286)	1 3 6 10 9 3 3	287)	1 3 6 10 9 4 3	288)	1 3 6 10 9 5 3
289)	1 3 6 10 9 6 3	290)	1 3 6 10 9 7 3	291)	1 3 6 10 9 8 3
292)	1 3 6 10 9 9 3	293)	1 3 6 10 10 3 3	294)	1 3 6 10 10 4 3
295)	1 3 6 10 10 5 3	296)	1 3 6 10 10 6 3	297)	1 3 6 10 10 7 3
298)	1 3 6 10 10 8 3	299)	1 3 6 10 10 9 3	300)	1 3 6 10 11 3 3
301)	1 3 6 10 11 4 3	302)	1 3 6 10 11 5 3	303)	1 3 6 10 11 6 3
304)	1 3 6 10 11 7 3	305)	1 3 6 10 11 8 3	306)	1 3 6 10 11 9 3
307)	1 3 6 10 12 3 3	308)	1 3 6 10 12 4 3	309)	1 3 6 10 12 5 3

310)	1 3 6 10 12 6 3	311)	1 3 6 10 12 7 3	312)	1 3 6 10 12 8 3
313)	1 3 6 10 12 9 3	314)	1 3 6 10 13 3 3	315)	1 3 6 10 13 4 3
316)	1 3 6 10 13 5 3	317)	1 3 6 10 13 6 3	318)	1 3 6 10 13 7 3
319)	1 3 6 10 13 8 3	320)	1 3 6 10 13 9 3	321)	1 3 6 10 14 3 3
322)	1 3 6 10 14 4 3	323)	1 3 6 10 14 5 3	324)	1 3 6 10 14 6 3
325)	1 3 6 10 14 7 3	326)	1 3 6 10 14 8 3	327)	1 3 6 10 14 9 3
328)	1 3 6 10 15 3 3	329)	1 3 6 10 15 4 3	330)	1 3 6 10 15 5 3
331)	1 3 6 10 15 6 3	332)	1 3 6 10 15 7 3	333)	1 3 6 10 15 8 3
334)	1 3 6 10 15 9 3				

TABLE 9

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ABSTRACT

Non Existence of Artinian Level Algebra of codimension 3 and type 3

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We try to classify Artinian \mathcal{O} -sequences of codimension 3 and type 3 with length 7 based on whether or not being level. In this thesis, we shall prove that some \mathcal{O} -sequences among the above 334 cases, which look like possible level Artinian \mathcal{O} -sequences and produce some of them using 'Link-sum' constructions of sets of points in P^2 .

However 68 cases are still open which we should prove.