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Local Maxima and Local Minima of Non-Unimodal Artinian O-sequences

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성신여자대학교 교육대학원

교육학과 수학교육전공

이 혜 진

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이 논문을 석사학위논문으로 제출함.

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인 준 서

이혜진의 석사학위 논문으로 인준함.

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논 문 개 요

[10]에서 F. Zanello는 여차원이 3인 단일 양상이 아닌 Level O-수열(Non-Unimodal Level O-sequence)에 관하여 다음과 같은 Zanello의 구조를 발견하였는데,

$$..., t, t, t+1, t, t, t+1, ...,$$

이 구조는 원하는 만큼의 극대값(Maxima)을 가진다.

[9]에서 Y. S. Shin은 단일 양상이 아닌 Level O-수열이 원하는 만큼의 다른 국소적 극대값(Local Maxima)을 만들 수 있는 방법을 보였다. 특히 그 극대값들은 모두 증가하거나 감소한다.

우리는 위의 논문을 토대로 여차원이 3인 단일 양상이 아닌 Level O-수열의 국소적 극대값들은 어떤 조건하에 감소하는 행동양식을 보인다는 것을 증명하였으며, 이 때 국소적 극소값(Local Minima)은 같은 차원을 같은 두 값이 나타남을 발견하였다. 또한 어떤 조건하에 국소적 극소값들은 국소적 극대값과 마찬가지로 감소함을 증명하였다.

이 논문에서 모든 예제는 컴퓨터 프로그램 CoCoA의 알고리즘이 사용되었다. 사용된 알고리즘은 D.J. Lee와 공동작업하였다.

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ABSTRACT

1. Introduction

If we let $R = k[x_0, x_1, \ldots, x_n] = \oplus_{i\geq 0} R_i$, where k is an algebraically closed field of characteristic 0, and let I be a homogeneous ideal of R, $A = R/I$, then the *Hilbert function of* A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

In the case I is the ideal of a subscheme $\mathbb{X}$ of $\mathbb{P}^n$, the Hilbert function of $A = R/I$ is sometimes denoted by $\mathbf{H}_{\mathbb{X}}(-)$.

We associate to the graded Artinian algebra A a vector of non-negative integers which is an $(s+1)$-tuple, called the h-vector of A and denoted $h(A)$. Let A be as above. Then it is defined as follows.

$$h(A) := (1, \dim_k A_1, \ldots, \dim_k A_s) = (h_0, h_1, \ldots, h_s) \quad \text{with} \quad h_s \neq 0.$$

Let $\mathcal{F}$ be the minimal free resolution of R/I, i.e.,

$$\mathcal{F} : 0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to R \to R/I \to 0.$$

We can write

$$\mathcal{F}_i = \bigoplus_{j=1}^{\gamma_i} R^{\beta_{ij}}(-\alpha_{ij})$$

where $\alpha_{i1} < \cdots < \alpha_{i\gamma_i}$. The numbers α_{ij} are called the *shifts* associated to R/I, and the numbers β_{ij} are called the *graded Betti numbers* of R/I.

Now we recall that if the last free module of the minimal free resolution of a graded ring A with Hilbert function $\mathbf{H}$ is of the form $R^m(-s)$ for some s, then the Hilbert function $\mathbf{H}$ and a graded ring A are called *level* (see [1],[2],[3],[4], [6], [7], [8],[9]).

In [10], F. Zanello found how to construct a non-unimodal level O-sequence which is ending as follows: $\ldots, t, t, t+1, t, t, t+1, \ldots,$ and a very specific pattern of local maxima. In [9], he showed how to produce a non-unimodal level O-sequence which has different local maxima as many as we desire. In particular, they are all either increasing or decreasing.

The goal of this paper is to find the behavior of local maxima of a codimension 3 non-unimodal level O-sequence which has the decreasing local maxima (see Theorem 2.6). Moreover, we prove that the local minima of some codimension 3 non-unimodal level O-sequence exist in two consecutive degrees (see Theorem 2.4) and its local minima are decreasing with a certain condition (see Theorem 2.7).

The computer program CoCoA [5] was used for all examples in this article.

2. Local Maxima and Local Minima of a Non-Unimodal Level O-sequence

First, we introduce the theorem of Iarrobino to construct a level O-sequence from the given level O-sequence.

Theorem 2.1 (Theorem 4.8A, [5])**.** *Let* $\mathbf{H}' = (h_0, h_1, \dots, h_s)$ *be the h-vector of a level algebra* $A = R/\mathrm{Ann}(M)$*. Then, if* F *is a generic form of degree* s*, the level algebra* $A = R/\mathrm{Ann}(\langle M, F\rangle)$ *has the h-vector* $\mathbf{H} = (H_0, H_1, \dots, H_s)$ *where, for* $i = 1, \dots, s$*,*

$$H_i = \min\left\{ h_i + \binom{r-1+s-i}{s-i}, \binom{r-1+i}{i} \right\}.$$

The following algorithm is to produce a level Artinian O-sequence from the given level O-sequence based on Theorem 2.1.

Algorithm 2.2. (CoCoA).

```
Define ADDUPHilbert(L)
        S := [1];
    For I := 2 To Len(L) Do
        S1 := Sum(First(L,I));
        Append(S, S1)
    EndFor;
    S2 := Comp(S, Len(S));
    Return S;
EndDefine;


Define LEVELHVECTOR(T)
    NewT := [1];
    R := Comp(T,2);
    E := Len(T)-1;
    For J := 2 To Len(T) Do
        I := J-1;
        Ti := Comp(T,J);
        T1 := Bin(R-1+E-I, E-I);
        T2 := Bin(R-1+I, I);
        NewTi := Min(Ti+T1, T2);
        Append(NewT, NewTi);
    EndFor;
    Return NewT;
EndDefine;


Define MAKELocalSeq(Seq, A, B, N)
    NewN := B;
    For I := 1 To N Do
        NewN := NewN * A;
```

```
    EndFor;
    For I := 1 To NewN Do
        Append(Seq, NewN);
    EndFor;
    Return Seq;
EndDefine;


Define MAKESequence(S, A, B, N)
    Seq := [];
    For I := 1 To S Do
        Append(Seq, I)
    EndFor;
    For I := 0 To N Do
        Seq := MAKELocalSeq(Seq, A, B, N-I);
    EndFor;
    Return Seq;
EndDefine;


Define PRINTMaxima(NewOSeq, A, B, N)
    Print "T = ", NewLine, NewOSeq, NewLine;
    L := Len(NewOSeq);
    MinimaSeq := [];
    MaximaSeq := [];
    IsIncrease := 1;
    Max := -1;  Min := -1;  Temp := -1;
    For I := 1 To L Do
        Temp := Comp(NewOSeq, I);
        If IsIncrease = 1   Then
            If Temp >= Max  Then
```

```
                Max := Temp;
            Else
                Append(MaximaSeq, Max);
                Min := Max;
                IsIncrease := 0;
            EndIf;
        Else
            If Temp <= Min  Then
                Min := Temp;
            Else
                Append(MinimaSeq, Min);
                Max := Min;
                IsIncrease := 1;
            EndIf;
        EndIf;
    EndFor;
    TempSeq := MaximaSeq;
    MaximaSeq := [];
    For I := Len(TempSeq) - N +1 To Len(TempSeq) Do
        Append(MaximaSeq, Comp(TempSeq, I));
    EndFor;
    Print NewLine, NewLine, "Maxima = ", MaximaSeq, NewLine;
EndDefine;


Define DOJob(S, A, B, N)
    Seq := MAKESequence(S, A, B, N);
    Print Seq, NewLine, NewLine;
    NewH := ADDUPHilbert(Seq);
    Print NewH, NewLine, NewLine;
    NewOSeq := LEVELHVECTOR(NewH);
```

```
    PRINTMaxima(NewOSeq, A, B, N);
EndDefine;
```

Here is an example of level Artinian O-sequence which can be obtained using Algorithm 2.2.

Example 2.3 (CoCoA)**.** Consider an O-sequence $\mathbf{H}'$ of codimension 3 such that

$$\Delta\mathbf{H}' = (1,2,3,\ldots,300,\underbrace{32,32,\ldots,32}_{\text{32-times}},\underbrace{16,16,\ldots,16}_{\text{16-times}},\underbrace{8,8,\ldots,8}_{\text{8-times}},4,4,4,4).$$

Using the command DOJob(S,A,B,N) from CoCoA with S=300,A=2,B=4,and N=3, we obtain a non-unimodal level O-sequence $\mathbf{H}$ of codimension 3 having 4 local maxima as follows.

$$\begin{aligned}\mathbf{H} = {} & (1,3,6,10,15,21,28,36,45,\ldots,46621,46616,46612,\\ & 46609,46607,46606,46606,46607,\mathbf{46609},46596,46584,46573,\\ & 46563,46554,46546,46539,46533,46528,46524,46521,\\ & 46519,46518,46518,46519,\mathbf{46521},46516,46512,46509,\\ & 46507,46506,46506,46507,\mathbf{46509},46508,46508,46509,46511).\end{aligned}$$

In fact, Algorithm 2.2 with the command DOJob(300,2,4,3)from CoCoA produces a level O-sequence $\mathbf{H}$ using the given level O-sequence.

In the ezample we can see that the local maxima are decreasing. In fact, we will show that it is always true under some additional conditions(see Theorem 2.6).

Theorem 2.4. *Let a and b are positive integers such that $a = 2$ and $b \geq 3$, and let*

$$\begin{aligned}\Delta\mathbf{H}' &= (1, 2, 3, \ldots, s+1, \\ &\quad \underbrace{a^N \cdot b, \ldots, a^N \cdot b}_{a^N \cdot b\text{-times}}, \underbrace{a^{N-1} \cdot b, \ldots, a^{N-1} \cdot b}_{a^{N-1} \cdot b\text{-times}}, \\ &\quad \vdots \\ &\quad \underbrace{a^2 \cdot b, \ldots, a^2 \cdot b}_{a^2 \cdot b\text{-times}}, \underbrace{a \cdot b, \ldots, a \cdot b}_{a \cdot b\text{-times}}, \underbrace{b, \ldots, b}_{b\text{-times}})\end{aligned}$$

where $s \gg 0$. Let $\mathbf{H}$ be a level O-sequence obtained from Theorem 2.1 with $\mathbf{H}'$. If $a = 2$, $b \geq 3$, then local minima of $\mathbf{H}$ exist in two consecutive degrees.

Proof. Let

$$\alpha_\ell = s + 2^{N+1} \cdot b + 2^N \cdot b + \cdots + 2^{N-\ell+1} \cdot b,$$

First we shall show that

$$H_{\alpha_\ell + 2^{N-\ell} \cdot b - b + 1} = H_{\alpha_\ell + 2^{N-\ell} \cdot b - b + 2}.$$

Since

$$\begin{aligned}&H_{\alpha_\ell + 2^{N-\ell} \cdot b - b + 1} \\ = \;& \binom{s+2}{2} + \underbrace{2^{N+1} \cdot b + \cdots + 2^{N+1} \cdot b}_{2^{N+1} \cdot b\text{-times}} + \cdots + \\ & \underbrace{2^{N-\ell} \cdot b + \cdots + 2^{N-\ell} \cdot b}_{2^{N-\ell} \cdot b - b + 1\text{-times}} + \binom{2^{N-\ell} \cdot b + 1}{2^{N-\ell} \cdot b - 1},\end{aligned}$$

and

$$\begin{aligned} & H_{\alpha_\ell+2^{N-\ell}\cdot b-b+2} \\ = \; & \binom{s+2}{2} + \underbrace{2^{N+1}\cdot b+\cdots+2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\ & \underbrace{2^{N-\ell}\cdot b+\cdots+2^{N-\ell}\cdot b}_{2^{N-\ell}\cdot b-b+2\text{-times}} + \binom{2^{N-\ell}\cdot b}{2^{N-\ell}\cdot b-2}, \end{aligned}$$

we have

$$\begin{aligned} & H_{\alpha_\ell+2^{N-\ell}\cdot b-b+1} - H_{\alpha_\ell+2^{N-\ell}\cdot b-b+2} \\ = \; & \left[\underbrace{2^{N-\ell}\cdot b+\cdots+2^{N-\ell}\cdot b}_{2^{N-\ell}\cdot b-b+1\text{-times}} + \binom{2^{N-\ell}\cdot b+1}{2^{N-\ell}\cdot b-1} \right] - \\ & \left[\underbrace{2^{N-\ell}\cdot b+\cdots+2^{N-\ell}\cdot b}_{2^{N-\ell}\cdot b-b+2\text{-times}} - \binom{2^{N-\ell}\cdot b}{2^{N-\ell}\cdot b-2} \right] \\ = \; & -2^{N-\ell}\cdot b + \tfrac{1}{2}(2^{N-\ell}\cdot b+1)(2^{N-\ell}\cdot b) - \tfrac{1}{2}(2^{N-\ell}\cdot b)(2^{N-\ell}\cdot b-1) \\ = \; & -2^{N-\ell}\cdot b + 2^{N-\ell}\cdot b \\ = \; & 0, \end{aligned}$$

which means that

$$H_{\alpha_\ell+2^{N-\ell}\cdot b-b+1} = H_{\alpha_\ell+2^{N-\ell}\cdot b-b+2}.$$

Now we will show that $H_{\alpha_\ell+2^{N-\ell}\cdot b-b+1}$ and $H_{\alpha_\ell+2^{N-\ell}\cdot b-b+2}$ are local minima.

In other words, it suffices to show

$$H_{\alpha_\ell} > \cdots > H_{\alpha_\ell+2^{N-\ell}\cdot b-b+1} = H_{\alpha_\ell+2^{N-\ell}\cdot b-b+2} < \cdots < H_{\alpha_{\ell+1}}.$$

First, we show that

$$H_{\alpha_\ell+i-1} > H_{\alpha_\ell+i}, \qquad 1 \le i \le 2^{N-\ell}\cdot b - b + 1.$$

Since

$$\begin{aligned} & H_{\alpha_\ell+i-1} \\ = & \binom{s+2}{2} + \underbrace{2^{N+1}\cdot b + \cdots + 2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\ & \underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{i-1\text{-times}} + \binom{2^{N-\ell+1}\cdot b - b + 2 - (i-1)}{2^{N-\ell+1}\cdot b - b - (i-1)}, \end{aligned}$$

and

$$\begin{aligned} & H_{\alpha_\ell+i} \\ = & \binom{s+2}{2} + \underbrace{2^{N+1}\cdot b + \cdots + 2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\ & \underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{i\text{-times}} + \binom{2^{N-\ell+1}\cdot b - b + 2 - i}{2^{N-\ell+1}\cdot b - b - i}, \end{aligned}$$

we have

$$
\begin{aligned}
& H_{\alpha_\ell+i-1} - H_{\alpha_\ell+i} \\
= & \left[\underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{i-1\text{-times}} + \binom{2^{N-\ell+1}\cdot b - b + 2 - (i-1)}{2^{N-\ell+1}\cdot b - b - (i-1)} \right] - \\
& \left[\underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{i\text{-times}} - \binom{2^{N-\ell+1}\cdot b - b + 2 - i}{2^{N-\ell+1}\cdot b - b - i} \right] \\
= & -2^{N-\ell}\cdot b + \tfrac{1}{2}(2^{N-\ell+1}\cdot b - b + 3 - i)(2^{N-\ell+1}\cdot b - b + 2 - i) \\
& -\tfrac{1}{2}(2^{N-\ell+1}\cdot b - b + 2 - i)(2^{N-\ell+1}\cdot b - b + 1 - i) \\
= & -2^{N-\ell}\cdot b + 2^{N-\ell+1}\cdot b + \tfrac{1}{2}(-2b - 2i + 4) \\
= & -2^{N-\ell}\cdot b + 2\cdot 2^{N-\ell}\cdot b - b - i + 2 \\
= & \; b(2^{N-\ell} - 1) + 2 - i \\
\geq & \; b(2^{N-\ell} - 1) + (2 - 2^{N-\ell}\cdot b - b + 1) \qquad (\because 1 \leq i \leq 2^{N-\ell}\cdot b - b + 1) \\
= & \; 1 \\
> & \; 0,
\end{aligned}
$$

which shows that

$$H_{\alpha_\ell+i-1} > H_{\alpha_\ell+i},$$

for every $i = 1, 2, \ldots, 2^{N-\ell}\cdot b - b + 1$.

Now we will show that

$$H_{\alpha_\ell+j-1} < H_{\alpha_\ell+j}, \qquad 2^{N-\ell}\cdot b - b + 3 \le j \le 2^{N-\ell}\cdot b.$$

Since

$$\begin{aligned} & H_{\alpha_\ell+j-1} \\ = \ & \binom{s+2}{2} + \underbrace{2^{N+1}\cdot b + \cdots + 2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\ & \underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{j-1\text{-times}} + \binom{2^{N-\ell+1}\cdot b - b + 2 - (j-1)}{2^{N-\ell+1}\cdot b - b - (j-1)}, \end{aligned}$$

and

$$\begin{aligned} & H_{\alpha_\ell+j} \\ = \ & \binom{s+2}{2} + \underbrace{2^{N+1}\cdot b + \cdots + 2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\ & \underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{j\text{-times}} + \binom{2^{N-\ell+1}\cdot b - b + 2 - j}{2^{N-\ell+1}\cdot b - b - j}, \end{aligned}$$

we have

$$\begin{aligned} & H_{\alpha_\ell+j} - H_{\alpha_\ell+j-1} \\ = \ & \left[\underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{j\text{-times}} + \binom{2^{N-\ell+1}\cdot b - b + 2 - j}{2^{N-\ell+1}\cdot b - b - j} \right] - \\ & \left[\underbrace{2^{N-\ell}\cdot b + \cdots + 2^{N-\ell}\cdot b}_{j-1\text{-times}} - \binom{2^{N-\ell+1}\cdot b - b + 2 - (j-1)}{2^{N-\ell+1}\cdot b - b - (j-1)} \right] \end{aligned}$$

$$
\begin{aligned}
&= 2^{N-\ell} \cdot b + \tfrac{1}{2}(2^{N-\ell+1} \cdot b - b + 2 - j)(2^{N-\ell+1} \cdot b - b + 1 - j) - \\
&\quad \tfrac{1}{2}(2^{N-\ell+1} \cdot b - b + 3 - j)(2^{N-\ell+1} \cdot b - b + 2 - j) \\
&= 2^{N-\ell} \cdot b - 2^{N-\ell+1} \cdot b + b - 2 + j \\
&= -2^{N-\ell} \cdot b + b - 2 + j \\
&\geq -2^{N-\ell} \cdot b + b - 2 + 2^{N-\ell} \cdot b - b + 3 \\
&\quad (\because \quad 2^{N-\ell} \cdot b - b + 3 \leq j \leq 2^{N-\ell} \cdot b) \\
&= 1 \\
&> 0,
\end{aligned}
$$

which means that

$$H_{\alpha_\ell + j - 1} < H_{\alpha_\ell + j}, \qquad \text{for every } j = 2^{N-\ell} \cdot b - b + 3, \ldots, 2^{N-\ell} \cdot b.$$

Therefore, we have that

$$H_{\alpha_\ell} > \cdots > H_{\alpha_\ell + 2^{N-\ell} . b - b + 1} = H_{\alpha_\ell + 2^{N-\ell} . b - b + 2} < \cdots < H_{\alpha_{\ell+1}},$$

as we desired. □

Corollary 2.5. *Let* **H** *be as in Theorem 2.4. Then* **H** *has the local maxima at degrees*

$$\alpha_\ell = s + 2^{N+1} \cdot b + s + 2^N \cdot b + \cdots + 2^{N-\ell+1} \cdot b, \quad \ell = 0, 1, \cdots, N-1.$$

Proof. The result is immediate from the proof of Theorem 2.4. □

Theorem 2.6. *With notation as in Theorem 2.4,* **H** *is a level O-sequence of codimension* 3 *having* $N+1$ *local maxima. Furthermore,all* N*-local maxima are decreasing for* $a = 2$, $b \geq 3$.

Proof. We will show that

$$H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell+1}\cdot b} > H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell}\cdot b}$$

for every $\ell = 0, 1, \ldots, N-1$.

In fact, by Theorem 2.4, we have

$$\begin{aligned} & H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell+1}\cdot b} \\ = \; & \binom{s+2}{s} + \underbrace{2^{N+1}\cdot b + \cdots + 2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\ & \underbrace{2^{N-\ell+1}\cdot b + \cdots + 2^{N-\ell+1}\cdot b}_{2^{N-\ell+1}\cdot b\text{-times}} + \binom{2^{N-\ell+1}\cdot b - b + 2}{2^{N-\ell+1}\cdot b - b}, \end{aligned}$$

and

$$
\begin{aligned}
& H_{s+2^{N+1}\cdot b+2^{N}\cdot b+\cdots+2^{N-\ell}\cdot b} \\
= & \binom{s+2}{s} + \underbrace{2^{N+1}\cdot b+\cdots+2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}}+\cdots+ \\
& \underbrace{2^{N-\ell}\cdot b+\cdots+2^{N-\ell}\cdot b}_{2^{N-\ell}\cdot b\text{-times}}+\binom{2^{N-\ell}\cdot b-b+2}{2^{N-\ell}\cdot b-b},
\end{aligned}
$$

and hence

$$
\begin{aligned}
& H_{s+2^{N}\cdot b+2^{N-1}\cdot b+\cdots+2^{N-\ell+1}\cdot b} - H_{s+2^{N}\cdot b+2^{N-1}\cdot b+\cdots+2^{N-\ell}\cdot b} \\
= & \binom{2^{N-\ell+1}\cdot b-b+2}{2^{N-\ell+1}\cdot b-b} - (b\cdot 2^{N-\ell})^2 - \binom{2^{N-\ell}\cdot b-b+2}{2^{N-\ell}\cdot b-b} \\
= & \tfrac{1}{2}(b\cdot 2^{N-\ell+1}-b+2)(b\cdot 2^{N-\ell+1}-b+1)- \\
& (b\cdot 2^{N-\ell})^2 - \tfrac{1}{2}(b\cdot 2^{N-\ell}-b+2)(b\cdot 2^{N-\ell}-b+1) \\
= & \tfrac{1}{2}\cdot b\cdot(b\cdot 2^{N-\ell}-2b+3) \\
= & \tfrac{1}{2}\cdot b\cdot\Big[b(2^{N-\ell}\cdot b-2)+3\Big] \\
> & 0 \quad (\because N-\ell\geq 1),
\end{aligned}
$$

which follows

$$
H_{s+2^{N}\cdot b+2^{N-1}\cdot b+\cdots+2^{N-\ell+1}\cdot b} > H_{s+2^{N}\cdot b+2^{N-1}\cdot b+\cdots+2^{N-\ell}\cdot b}
$$

for every $\ell = 0,1,\ldots,N-1$, as we wished. □

Theorem 2.7. *Let* **H** *be as in Theorem 2.4. Then all N-local minima are decreasing.*

Proof. By Theorem 2.4, it suffices to show that

$$H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell+1}\cdot b-b+1} > H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell}\cdot b-b+1}$$

for every $\ell = 0, 1, 2, \cdots, N-1$.

Since

$$\begin{aligned}
& H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell+1}\cdot b-b+1} \\
= & \binom{s+2}{s} + \underbrace{2^{N+1}\cdot b+\cdots+2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\
& \underbrace{2^{N-\ell+1}\cdot b+\cdots+2^{N-\ell+1}\cdot b}_{2^{N-\ell+1}\cdot b-b+1\text{-times}} + \binom{2^{N-\ell+1}\cdot b+1}{2^{N-\ell+1}\cdot b-1},
\end{aligned}$$

and

$$\begin{aligned}
& H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell}\cdot b-b+1} \\
= & \binom{s+2}{s} + \underbrace{2^{N+1}\cdot b+\cdots+2^{N+1}\cdot b}_{2^{N+1}\cdot b\text{-times}} + \cdots + \\
& \underbrace{2^{N-\ell}\cdot b+\cdots+2^{N-\ell}\cdot b}_{2^{N-\ell}\cdot b-b+1\text{-times}} + \binom{2^{N-\ell}\cdot b+1}{2^{N-\ell}\cdot b-1},
\end{aligned}$$

we have

$$\begin{aligned}
& H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell+1}\cdot b-b+1} - H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell}\cdot b-b+1} \\
= & \binom{2^{N-\ell+1}\cdot b+1}{2^{N-\ell+1}\cdot b-1} - \binom{2^{N-\ell}\cdot b+1}{2^{N-\ell}\cdot b-1} - (2^{N-\ell}\cdot b)(2^{N-\ell}\cdot b-b+1) \\
& -(b-1)(2^{N-\ell+1}\cdot b)
\end{aligned}$$

$$
\begin{aligned}
&= 2(2^{N-\ell}\cdot b)^2 + 2^{N-\ell+1}\cdot b) - b\cdot(2^{N-\ell}\cdot b)\\
&= (2^{N-\ell}\cdot b)\big[b(2^{N-\ell}-1)+2\big]\\
&> 0 \quad (\because N-\ell\ge 1, b\ge 3),
\end{aligned}
$$

which follows that

$$H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell+1}\cdot b-b+1} > H_{s+2^{N+1}\cdot b+2^N\cdot b+\cdots+2^{N-\ell}\cdot b-b+1}$$

for every $\ell = 0,1,\ldots,N-1$, as we wished. $\square$

Example 2.8 (CoCoA)**.** Consider an O-sequence $\mathbf{H}'$ of codimension 3 such that

$$\Delta\mathbf{H}' = (1,2,3,\ldots,300,\underbrace{32,32,\ldots,32}_{\text{32-times}},\underbrace{16,16,\ldots,16}_{\text{16-times}},\underbrace{8,8,\ldots,8}_{\text{8-times}},4,4,4,4).$$

With the command DOJob(S,A,B,N) from CoCoA with S=300,A=2,B=4,and N=3, we obtain a non-unimodal level O-sequence $\mathbf{H}$ of codimension 3 having 4 local minima as follows.

$$
\begin{aligned}
\mathbf{H} = (&1,3,6,10,15,21,28,36,45,\ldots,46621,46616,46612,\\
&46609,46607,\mathbf{46606},\mathbf{46606},46607,46609,46596,46584,46573,\\
&46563,46554,46546,46539,46533,46528,46524,46521,\\
&46519,\mathbf{46518},\mathbf{46518},46519,46521,46516,46512,46509,\\
&46507,\mathbf{46506},\mathbf{46506},46507,46509,46508,46508,46509,46511),
\end{aligned}
$$

which shows that the local minima of $\mathbf{H}$ are decreasing.

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ABSTRACT

Local Maxima and Local Minima of Non-Unimodal Artinian O-sequences

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In [10], F. Zanello found how to construct a non-unimodal level O-sequence which is ending as follows: ..., t, t, $t+1$, t, t, $t+1$, ..., and a very specific pattern of local maxima.

In [9], he showed how to produce a non-unimodal level O-sequence which has different local maxima as many as we desire. In particular, they are all either increasing or decreasing.

The goal of this paper is to find the behavior of local maxima of a codimension 3 non-unimodal level O-sequence which has the decreasing local maxima. Moreover, we prove that the local minima of some codimension 3 non-unimodal level O-sequence exist in two consecutive degree and its local minima are decreasing with a certain condition.

A computer program CoCoA was used for all example in this thesis.