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碩士學位 請求論文

Fixed Points of Increasing Operators on Ordered Spaces

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誠信女子大學校 教育大學院

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목 차

논 문 개 요

순서집합(順序集合 ; ordered set)에서 증가연산자(增加演算子 ; increasing operator)의 부동점(不動點 ; fixed point)정리가 Zorn의 보조정리와 연관이 있으며, 극대원의 존재성과 밀접한 관계가 있음은 잘 알려진 사실이다.

1996년에 Fang [6]은 위로 순서완비반속(順序完備半束 ; order complete upper-semilattice)이 상한점열적(上限點列的 ; supremum sequentiable)일 때, 그 위에서 정의된 증가함수의 부동점정리를 증명하였고, 이것이 적분방정식의 해를 구하는 데 응용될 수 있음을 보였다.

이 논문에서 우리는 Fang과는 달리 일반 순서집합에서 상한점열적이라는 조건을 가정하지 않고 증가함수의 부동점정리가 성립함을 보였다. 또 순서가분거리공간 (順序可分距離空間 ; ordered separable metric space)이 상한점열적임을 보이고, 이를 이용하여 이 공간이 점별순서콤팩트(sequentially order compact)이면 완비반속(完備半束 ; complete semilattice)임을 보였다.

이 논문에서는 또 점별순서콤팩트인 순서가분집합에서의 증가함수의 부동점정리를 얻었다. 또한, 순서위상공간에서 혼합단조연산자(混合單調演算子 ; mixed monotone operator)의 부동점정리를 증명하였다.

I. Introduction

A *quasi-ordered set* $(X, \leq)$ is a nonempty set together with a reflexive and transitive relation $\leq$ on X. If $\leq$ is antisymmetric, then $(X, \leq)$ is called a *partially ordered set*.

The following Zorn's lemma is valid for quasi-ordered sets.

Theorem (Zorn's Lemma). *Let $(X, \leq)$ be a quasi-ordered set. If every chain in X has an upper bound, then X has a maximal element.*

Consider the following conditions;

(C1) Every nonempty chain in X has a supremum.

(C2) Every nonempty chain in X has an upper bound.

(W1) Every nonempty well-ordered subset of X has a supremum.

(W2) Every nonempty well-ordered subset of X has an upper bound.

Note that in the theorem above the condition (C2) is assumed for Zorn's lemma.

It is well-known that Zorn's lemma is valid under each assumption (C1), (W1) or (W2) instead of (C2). There have been a lot of maximal principles using (C1)(See [7]), (W1)(See [12]) or (W2)(See [1]).

Also it is known that Zorn type maximal principles are equivalent to some fixed point theorems for functions on ordered sets.

The following fixed point theorem is equivalent to Zorn's Lemma (See Dunford and Schwartz [5]).

Theorem (Zermelo). *Let X be a partially ordered set and assume that every chain in X has a supremum. Then every selfmap $f : X \to X$ satisfying*

$$x \leq f(x), \quad x \in X$$

has a fixed point.

In 1955, Tarski proved the following fixed point theorem for increasing operators on complete lattices;

Theorem (Tarski). *Let X be a complete lattice. Then every increasing selfmap $f : X \to X$ has a fixed point.*

Park [10] showed that Zorn's lemma can be reformulated to some fixed point theorems on ordered sets. Also he obtained some fixed point theorems for multivalued functions on ordered spaces [11].

Fixed point theorems for increasing operators in quasi-ordered sets have wide applications in topology and nonlinear analysis.

In 1996, Fang [6] proved a fixed point theorem for increasing operators in a supremum sequentiable and ordered complete upper-semilattice. Also he showed that his theorem can be applied to solve some integral equations.

In this paper, we will discuss some fixed point theorems for increasing operators on ordered spaces.

Section II deals with some basic definitions and preliminary results concerning ordered sets, supremum sequentiable spaces and others.

In Section III, as main results, we will prove a fixed point theorem in an order complete upper-semilattice without assuming supremum sequentiability. Also we will prove that an ordered separable metric space is supremum sequentiable.

By using them, we obtain a fixed point theorem for increasing operators in ordered separable metric space which is sequentially order compact. Moreover, we will prove the existence of coupled fixed points of mixed monotone operators in ordered topological spaces.

II. Preliminaries

A *quasi-ordered set* $(X, \leq)$ is a nonempty set together with a reflexive and transitive relation $\leq$ on X. If $\leq$ is antisymmetric, then $(X, \leq)$ is called a *partially ordered set.*

Let $(X, \leq)$ be a quasi-ordered set and $x \in X$. We define

$$S(x) = \{y \in X \mid x \leq y\}$$

to be the *super set* of x. Also the set

$$I(x) = \{y \in X \mid y < x\}$$

is called the *initial segment* of X determinded by x.

A *totally ordered* subset A of a quasi-ordered set $(X, \leq)$ is a subset of X such that for any $a, b \in A$,

$$a \leq b \text{ or } b \leq a.$$

A totally ordered subset of X is also called a *chain.*

Let $(X, \leq)$ be a quasi-ordered set. A partially ordered subset W of X is called a *well-ordered set* if every nonempty subset of W has the least element, that is, if C is any nonempty subset of W, then there is an element $c \in C$ such that $c \leq x$ for all $x \in C$. If $x \in W$ and there is an element

$y \neq x$ such that $x \leq y$, then the set $\{y \in X \mid x < y\}$ has the least element, say $x+1$. It is called the *immediate successor* of x. If x has no immediate successor, then x is called the *last element*. It is well-known that every subset of a well-ordered set X is isomorphic to X or an initial segment of X.

If A is a subset of a quasi-ordered set $(X, \leq)$, an element $x \in X$ is called an *upper bound* (*lower bound,* resp.) of A if

$$\forall a \in A, \quad a \leq x \ (x \leq a, \text{resp.})$$

An element $u \in X$ is called a *supremum* of A and is denoted by $u = \sup A$ if u is an upper bound of A and for all upper bound x of A, $u \leq x$ holds. Similarly an element v is called an *infimum* of A and is denoted by $v = \inf A$ if l is a lower bound of A and for all lower bound x of A, $x \leq v$ holds.

Let $(X, \leq)$ be a quasi-ordered set. An element $a \in X$ is said to be *maximal* (*minimal*, resp.) if

$$a \leq x \Longrightarrow x \leq a \ \ (x \leq a \Longrightarrow a \leq x, \ \text{resp.})$$

for all $x \in X$. Note that if $\leq$ is a partial order, and if a is a maximal element, then $x \leq a$ implies that $x = a$.

Now consider the following definitions.

Definition II. 1. Let X be a quasi-ordered set. We say that X is *supremum-sequentiable* if for any arbitrary totally ordered subset $M \subseteq X$ which has the supremum, there exists a sequence $\{x_n\} \subset M$ such that

$$\sup M = \sup\{x_n\}.$$

Definition II. 2. Let X be a quasi-ordered set, S a nonempty subset of X. S is said to be a *complete upper-semilattice* (*complete lower-semilattice,* resp.) if any nonempty subset of S has a supremum (infimum, resp.) in S. S is a *complete lattice* if S is not only a complete upper-semilattice but also a complete lower-semilattice. S is an *order complete upper-semilattice* (*order complete lower-semilattice,* resp.) if any totally ordered nonempty subset of S has a supremum (infimum, resp.) in S. S is an order complete semilattice if S is not only an order complete upper-semilattice but also an order complete lower-semilattice.

Definition II. 3. Let $(X, \leq)$ be a partially ordered set. If X is endowed with a topology τ, then $(X, \leq)$ is called an *ordered topological space* if for all convergent sequences $\{x_n\}$ and $\{y_n\}$, $x_n \leq y_n$ for all $n = 1, 2, 3, \cdots$ implies that

$$\lim x_n \leq \lim y_n.$$

Remark. If $(X, \leq)$ is an ordered topological space, then for each $x \in X$, $S(x)$ is ordered sequentially closed, that is, if $\{x_n\}$ is an increasing sequence in $S(x)$ which converges to some $x_0 \in X$, then $x_0 \in S(x)$. If $(X, \leq)$ is an ordered metric space, then $S(x)$ is closed.

Definition II. 4. Let $(X, \leq)$ be an ordered topological space. X is said to be *sequentially order compact* if every increasing or decreasing sequence in X has a convergent subsequence.

Let X be a topological space. We say that X is *second countable* if it has a countable topological basis. Also we say that X is *separable* if it has a countable dense subset.

Note also that any second countable topological space is separable and any separable metric space is second countable (See [2]).

Remark. Note that a topological space X is second countable if and only if it has a countable basis for the set of all closed subsets of X. That is, there is a countable collection $\{C_i\}_{i=1}^{\infty}$ of closed subsets of X such that for any closed subset F of X, there is a subcollection $\{C_{i_j}\}_{j=1}^{\infty}$ of $\{C_i\}_{i=1}^{\infty}$ satisfying

$$F = \bigcap_{j=1}^{\infty} C_{i_j}.$$

III. Main results

In this section, we will prove fixed point theorems for increasing operators on ordered spaces. To begin with, we state and prove some lemmas.

Lemma 1. *Let X be a partially ordered set. Then X is supremum sequentiable if and only if for any totally ordered subset $M \subseteq X$ which has supremum, there is an increasing sequence $\{x_n\}$ in M such that*

$$\sup M = \sup\{x_n\}.$$

Proof. Assume that M is supremum sequentiable. Then there is a sequence $\{x_n\}$ in M such that

$$\sup M = \sup\{x_n\}.$$

Let $y_1 = x_1$ and $y_2 = \max\{y_1, x_2\}$, then $y_1 \leq y_2$. For $n \geq 1$, we denote $y_n = \max\{y_{n-1}, x_n\}$. Then it is obvious that y_n is increasing and $x_n \leq y_n$ for all $n \in \mathbb{N}$.

Let $\sup M = \sup\{x_n\} = a$, then $y_n = \max\{x_1, x_2, \cdots, x_n\} \leq a$ for all $n \in \mathbb{N}$. Hence a is an upper bound of $\{y_n\}$. Let $b \in X$ be any upper bound of $\{y_n\}$, then it is an upper bound of $\{x_n\}$, since $x_n \leq y_n$ for all $n \in \mathbb{N}$. This shows that $a \leq b$ and also that a is the supremum of $\{y_n\}$. Therefore

$$a = \sup M = \sup\{y_n\}.$$

Remark. It is not always true that every subset of a supremum sequentiable partially ordered set is supremum sequentiable.

Lemma 2. *Let* $(X, \leq)$ *be an ordered topological space and* $\{x_n\}_{n=1}^{\infty}$ *be an increasing sequence in* X*. Assume that* $\{x_n\}$ *has a convergent subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *and* $x_0 = \lim_{k\to\infty} x_{n_k}$*, then* $x_0 = \sup x_n$*.*

Proof. For each $k \in \mathbb{N}$, $\{x_{n_i}\}_{i=k}^{\infty} \subseteq S(x_{n_k})$ and hence

$$x_0 = \lim_{i\to\infty} x_{n_i} \in S(x_{n_k}),$$

since $S(x_{n_k})$ is ordered sequentially closed. So $x_0 \geq x_{n_k}$ and $k \leq n_k$ implies that $x_0 \geq x_{n_k} \geq x_k$.

Since k was arbitrary, x_0 is an upper bound of $\{x_n\}_{n=1}^{\infty}$. If y is an upper bound of $\{x_n\}$, $x_{n_k} \leq y$ for all $k \in \mathbb{N}$. Hence

$$x_0 = \lim_{k\to\infty} x_{n_k} \leq y.$$

Therefore, $x_0 = \sup x_n$.

Theorem 3. *Let* $(X, \leq)$ *be an ordered separable metric space. Then* X *is supremum sequentiable.*

Proof. Let $M \subseteq X$ be a chain which has the supremum. Since M is seperable, there exists a countable dense subset $\{x_n\}$ of M. We will show that

$$\sup M = \sup\{x_n\}.$$

Let $u = \sup M$. Then obviously, $x_n \leq u$ for all $n \in \mathbb{N}$. Hence u is an upper bound of $\{x_n\}$. If u is not the supremum of $\{x_n\}$, then there exists an upper bound u_0 of $\{x_n\}$ such that $u_0 < u$.

Since $u_0 < u = \sup M$, u_0 is not an upper bound of M. So there exists an $x \in M$ such that $x \nleq u_0$.

Since $\{x_n\}$ is dense in M, $x = \lim_{i\to\infty} x_{n_i}$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Since $x_{n_i} \leq u_0$ for all $i = 1, 2, 3, \cdots$, we have $x = \lim_{i\to\infty} x_{n_i} \leq u_0$, which contradicts the fact that $x \nleq u_0$.

Therefore,

$$u = \sup\{x_n\}.$$

If $(X, \leq)$ is also sequentially order compact, we have the following:

Theorem 4. *Let $(X, \leq)$ be an ordered separable metric space which is sequentially order compact. Then X is an order complete semilattice and supremum sequentiable.*

Proof. By Theorem 3, it remains to show that X is an order complete semilattice. Let M be a nonempty totally ordered subset of X. Since X is a separable metric space, so is M. So there exists a countable dense subset $\{x_n\}_{n=1}^{\infty}$ of M. As in the proof of Lemma 1, we may assume that $\{x_n\}$ is increasing. Since X is sequentially order compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.

Let $x_0 = \lim_{k\to\infty} x_{n_k}$, then $x_0 = \sup\{x_n\}$ by Lemma 2. Since $\{x_n\}$ is dense in M, for all $x \in M$, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}$ such that

$$x = \lim_{i\to\infty} x_{n_i}.$$

Since $x_0 \geq x_{n_i}$ for each $i \in \mathbb{N}$, $\quad x_0 \geq x$.

This shows that x_0 is an upper bound of M. If y is an upper bound of M, then it is also an upper bound of $\{x_n\}$. Hence $x_0 \leq y$ and so

$$x_0 = \sup M.$$

Similarly, we can show that M has the infimum.

This shows that X is an order complete semilattice.

Fang[6] proved a fixed point theorem in a supremum sequentiable order complete upper-semilattice. We prove the following theorem without assuming supremum sequentiability.

Theorem 5. *Let $(X, \leq)$ be a partially ordered set and $D \subseteq X$. Let $f : D \to D$ be an increasing function and assume that D or $f(D)$ is an order complete upper-semilattice.*
Suppose that there exists an element $x \in D$ such that

$$x \leq f(x).$$

Then f has a maximal fixed point.

Proof. If $f(D)$ is an order complete upper-semilattice, then we may consider $f : f(D) \to f(D)$. So we may assume that D is an order complete upper-semilattice.
Let $P = \{x \in D \mid x \leq f(x)\}$ and $\sum$ be the collection of all well-ordered subset M of P such that $f(x)$ is the immediate succesor of x if $x \neq f(x)$ and x is not the last element of M. Order $\sum$ by the relation $C \preceq D$ iff C is an initial segment of D.

Let $\mathcal{C}$ be a well-ordered subset of $\sum$ and let $M_0 = \bigcup \mathcal{C}$. We will show that $M_0 \in \sum$.

To show that M_0 is well-ordered, let A be a nonempty subset of M_0. Since $\mathcal{C}$ is well-ordered, the set

$$\{C \in \mathcal{C} \mid \mathrm{A} \cap \mathrm{C} \neq \emptyset\}$$

has the least element, say C_0. Then $A \cap C_0$ has the least element x_0.

For any $a \in A, a \in M_0 = \bigcup \mathcal{C}$ implies that $a \in A \cap C$ for some $C \in \mathcal{C}$. But then $C_0 \preceq C$, since C_0 is the least element of $\mathcal{C}$ such that $A \cap C_0 \neq \emptyset$. Since C_0 is the initial segment of C, $A \cap C_0$ is the initial segment of $A \cap C$. Hence $a_0 \leq a$.

This shows that a_0 is the least element of A. Therefore, M_0 is well-ordered.

Moreover, if $x \in M_0$ is not the last element of M_0, then $x \in C$ for some $C \in \mathcal{C}$. We may assume that x is not the last element of C. Hence $f(x)$ is the immediate successor of x in C_0. If there is an element $y \in M_0$ such that $x \leq y \leq f(x)$, then $y \in C_1$ for some C_1 in $\mathcal{C}$. Since C is well-ordered, $C \preceq C_1$ or $C_1 \preceq C$. In any case, $x, y \in C$ or $x, y \in C_1$ and since $f(x)$ is the immediate successor of x there, $y = x$ or $y = f(x)$. This shows that $f(x)$ is the immediate successor of x in M_0.

We have shown that $M_0 \in \sum$ and hence it is the supremum of $\mathcal{C}$.

By Zorn's lemma, $\sum$ has a maximal element, say C_0. Since D is order complete upper-semilattice, C_0 has a supremum.

Let $x_0 = \sup C_0$. We will show that x_0 is a maximal fixed point of f. If x_0 is not a fixed point of f, $C_0 \neq C_0 \cup \{x_0, f(x_0)\}$ and $C_0 \cup \{x_0, f(x_0)\} \in \sum$, which contradicts the maximality of C_0.

If x_0 is not a maximal fixed point, there is an $y \in X$ such that $x_0 < y$ and $y = f(y)$. But then $C_0 \cup \{y\} \in \sum$ and $C_0 \neq C_0 \cup \{y\}$, contradiction.

Hence f has a maximal fixed point x_0.

The following is Fang's theorem [6].

Theorem 6. *Let X be a partially ordered set, $D \subseteq X$ be a supremum sequentiable and complete upper-semilattice. Let $f : D \to D$ is defined as above. Then the set of all fixed points $F = \{x \mid f(x) = x\}$ is a complete upper-semilattice.*

Proof. It remains to prove that F is a complete upper-semilattice. We refer the reader to the proof given in Fang [6].

Using Theorem 5 and Theorem 6, we can prove the following;

Theorem 7. *Let X be a partially ordered set, $D \subseteq X$ a nonempty set, $A : D \to D$ an increasing (decreasing, resp.) operator.*
Suppose that there exists $u \in D$ satisfying

$$u \leq Au \quad (Au \leq u, \text{ resp.}),$$

and there exist several partially ordered sets $D_1, D_2, \ldots, D_n$ and increasing (decreasing, resp.) operators

$$B_i : D_{i-1} \to D_i \ \ (i = 1, 2, \ldots, n+1),$$

such that

$$A = B_{n+1}B_n \dots B_1,$$

where $D_0 = D, \quad D_{n+1} = A(D)$. *Moreover, assume that for some* $i \in I = \{0, 1, 2, \dots, n+1\}, \quad D_i$ *is an order complete upper-semilattice. Then* A *has a fixed point. Moreover, if one of* D_i *is supremum sequentiable, then the fixed point set is a complete upper-semilattice.*

Proof. Assume that $i = 0$ or $n + 1$. Then the result follows from Theorem 4. Let $0 < i < n + 1$. Let $B = B_iB_{i-1} \dots B_2B_1, \quad C = B_{n+1}B_n \dots B_{i+1}$. Then $A = CB$ and $A(D) \subset D, \quad C(D_i) \subset D, \quad BC : D_i \to D_i$.
By Theorem 4, BC has a fixed point x_0. That is, $BC(x_0) = x_0$. Hence

$$AC(x_0) = CBC(x_0) = C(x_0)$$

and so $C(x_0)$ is a fixed point of A.

As mentioned above, if one of D_i is supremum sequentiable, then the set of all fixed points of A is a complete upper-semilattice.

From Theorem 4 and Theorem 6, we have the following;

Theorem 8. *Let* $(X, \leq)$ *be an ordered separable metric space. If* $D \subseteq X$ *is sequentially order compact and* $f : D \to D$ *is an increasing function and there exists an element* $x \in D$ *such that* $x \leq f(x)$, *then the fixed point set* $F = \{x \mid f(x) = x\}$ *of* f *is nonempty and a complete upper-semilattice.*

Proof. By Theorem 4, D is a supremum sequentiable and order complete semilattice. Hence the conclusion follows from Theorem 6.

Note that Theorem 8 is an extension of Theorem 1 in [13], which is a fixed point theorem for increasing operators in ordered normed spaces.

Now we will prove a fixed point theorem for mixed monotone operators. Some fixed point theorems for mixed monotone operators were obtained by Guo and Lakshimikantham [3], Yong Sun [15] and many others.

Let $(X, \leq)$ be a partially ordered set and $D \subseteq X$ be an order complete upper-semilattice. An operator $A : D \times D \to X$ is said to be *mixed monotone* if $A(x, y)$ is increasing in x for each fixed $y \in D$ and decreasing in y for each fixed $x \in D$. That is,

$$A(x_1, y) \leq A(x_2, y) \quad \text{if } x_1 \leq x_2,$$

and

$$A(x, y_1) \geq A(x, y_2) \quad \text{if } y_1 \leq y_2.$$

If there exist $x, y \in D$ with $x \leq y$ such that

$$x = A(x, y) \text{ and } A(y, x) = y,$$

then (x, y) is called a *coupled fixed point* of A. Moreover, if $A(x, x) = x$, then a point $x \in D$ is a *fixed point* of A ([15]).

Theorem 9. *Let $(X, \leq)$ be a partially ordered set and $D \subseteq X$ be an order complete semilattice.*

Let $A : D \times D \to X$ be a mixed monotone operator. If there is a $(u, v) \in D \times D$ such that

$$u \leq v \text{ and } u \leq A(u, v) \leq v, \quad u \leq A(v, u) \leq v,$$

then A has a coupled fixed point.

Proof. Let us define a partial ordering $\leq$ in $X \times X$ by

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2, \quad y_1 \geq y_2.$$

We will show that $D \times D$ is an order complete upper-semilattice. Let P be a totally ordered subset of $D \times D$. Then the sets

$$P_1 = \{a \in D \mid (a, b) \in P \text{ for some } b \in D\}$$

and

$$P_2 = \{b \in D \mid (a, b) \in P \text{ for some } a \in D\}$$

are totally ordered subsets of D.

Since D is an order complete lattice, P_1 has a supremum $a_0 \in X$ and P_2 has an infimum $b_0 \in X$. For any $(a, b) \in P$, we have

$$a \leq a_0 \text{ and } b_0 \leq b.$$

This shows that $(a,b) \le (a_0,b_0)$ and so (a_0,b_0) is an upper bound of P_0.
For any upper bound (u,v) of P in $X \times X$ and for any $(a,b) \in P$,

$$(a,b) \le (u,v)$$

implies that $a \le u$ and $v \le b$.

Hence u is an upper bound of P_1 and v is a lower bound of P_2. So $a_0 \le u$, $v \le b_0$ and $(a_0,b_0) \le (u,v)$. This shows that $(a_0,b_0) = \sup P$ in $X \times X$.

Therefore, $D \times D$ is an order complete upper-semilattice.
We define a function $F : D \times D \to X \times X$ by

$$F(u,v) = (A(u,v), A(v,u)), \quad (u,v) \in D \times D.$$

If $(x_1,y_1) \le (x_2,y_2)$, then we have

$$A(x_1,y_1) \le A(x_1,y_2) \le A(x_2,y_2)$$

and

$$A(y_2,x_2) \le A(y_1,x_2) \le A(y_1,x_1),$$

which implies that $F(x_1,y_1) \le F(x_2,y_2)$. Hence F is an increasing operator. By Theorem 4, F has a fixed point $(x_0,y_0) \in D \times D$. It is obvious that (x_0,y_0) is a coupled fixed point of the mixed monotone operator A.

From the above theorem and Theorem 4, we obtain the following;

Theorem 10. *Let* $(X, \leq)$ *be an ordered separable metric space. Let* $D \subseteq X$ *be sequentially order compact and* $A : D \times D \to X$ *be a mixed monotone operator.*
If there is a $(u, v) \in D \times D$ *such that*

$$u \leq v \text{ and } u \leq A(u, v) \leq v, \quad u \leq A(v, u) \leq v,$$

then A *has a coupled fixed point.*

References

1. Bourbaki, *Sur le theorem de Zorn*, Arth. Math 2, (1950), 434–437.

2. F. H. Croom, *Principles of Topology*, Philadelphia: Saunders College Pub., (1989).

3. Dajun Guo and V. Lakshimikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, (1988).

4. J. Dugundji and A. Granas, *Fixed Point Theory*, Polish Scientific Publishers, Warszawa, (1982).

5. N. Dunford and J. Schwartz, *Linear operators Part I*, Interscience Pub (1958), 4–7.

6. S. Fang, *Semilattice structure of fixed point set for increasing operators*, Nonlinear Analysis, Theory, Methods & Applications, Vol.27, No.7 (1996), 793-796.

7. A. G. Hamilton, *Numbers, Sets and Axioms*, Cambridge university press, (1982).

8. B. G. Kang, *Fixed points of increasing operators and maximal principles*, 성신연구논문집, 37 (2001), 219-228.

9. M. A. Krasnoselskij, Je. A. Lifshits and A. V. Sobolev, *Positive Linear Systems*, Heldermann Verlag, Berlin, (1989).

10. S. Park, *Equvalent formulations of Zorn's lemma and other maximum principles,* J. Korea Soc. Math. Ed. 25 (1987), 19–24.

11. S. Park, *Fixed points of multimaps on ordered spaces*, Math. Sci. Res. Hot-Line 2 (9) (1998), 9–14.

12. R. E. Smithson, *Fixed points of order preserving multifunctions*, Proc. Amer. Math. Soc. 28(1), (1971), 304–310.

13. Sun Jingxian and Sun Yong, *Some fixed point theorems for increasing operators*, Applicable Anal. 23, (1987), 19–24.

14. A. Tarski, *A lattice theoretic fixpoint theorems and its applications*, Pacific J. Math., 5, (1955), 285–310.

15. Yong Sun, *A fixed point theorem for mixed monotone operators with applications*, J. Math. Anal. Appl. 156, (1991), 240–252.

Abstract

It is well-known that a fixed point theorem for increasing operators is deeply related to Zorn′s lemma or the existence of maximal elements on ordered sets.

In 1996, Fang [6] proved a fixed point theorem for increasing operators on ordered complete upper-semilattices which are supremum sequentiable. Also he showed that his theorem could be applied to solve some integral equations.

In this paper, we showed a fixed point theorem for increasing operators on general ordered sets without assuming supremum sequentiability. Also we proved that an ordered separable metric space is supremum sequentiable. By using the above, we showed that if an ordered separable metric space is sequentially order compact, then it is a complete semilattice.

Moreover, we proved a coupled fixed point theorem for mixed monotone operators in ordered topological spaces.