姜秉介 教授指導
碩士學位 請求論文

Fixed Points of Increasing Operators on Fuzzy Numbers

2006

誠信女子大學校 教育大學院
教育學科 數學教育專攻
李 敬 姬

Fixed Points of Increasing Operators on Fuzzy Numbers

姜秉介 教授指導

이 論文을 碩士學位 論文으로 提出함

2006 年 6 月

誠信女子大學校 教育大學院

教育學科 數學教育專攻

李 敬 姬

認 准 書

李 敬 姬의 碩士學位 論文을 認准함

審査委員 ________________ 印

審査委員 ________________ 印

審査委員 ________________ 印

2006 年 6 月

誠信女子大學校 教育大學院

목　　차

논 문 개 요

Fuzzy 집합의 이론은 Zadeh[18]에 의하여 소개된 이래로 많은 학자들에 의하여 그 풍부한 응용성이 입증되었고, 따라서 fuzzy집합에 관한 다른 구조들이 정의되고 여러 가지 성질들이 증명되었다.

최근에는 fuzzy 집합에 거리를 정의하여 fuzzy 거리공간(fuzzy metric space)를 만들고, fuzzy 함수의 부동점(不動點 - fixed point) 이론을 활발하게 연구하고 있다.

2004년에 Chang et. al.[2]는 실수축에 정의된 fuzzy 수(fuzzy number)의 집합을 거리공간(metric space)화 하고 또 여기에 순서관계(order relation)을 정의하여 증가함수의 부동점정리를 증명하고, 이를 fuzzy 방정식의 해의 존재를 증명하는 데 이용하였다.

이 논문에서는 일반순서집합에서 증가함수의 부동점의 존재정리를 증명하고, 이를 fuzzy 수의 공간에 활용하였다. 특히 연속인 condensing 함수의 부동점정리를 증명하였다.

또한 혼합단조연산자(mixed monotone operator)에 관한 부동점정리도 증명하였다.

I. Introduction

The fuzzy set theory introduced by Zadeh [18] in 1965 has emerged an interesting branch of pure and applied sciences. Since then, a lot of structures on fuzzy sets are obtained and many authors have developed the fuzzy sets and their applications.

In 1984, Kaleva and Seikkala [12] introduced the concept of fuzzy metric space which generalizes the notion of metric space by setting the distance between two points to be a nonnegative fuzzy number, and investigated some connections between fuzzy metric spaces and probabilistic metric space.

Recently, George and Veeramani[7] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [14] and defined the Hausdorff topology of fuzzy metric spaces.

Heilpern [11] proved some Banach type fixed point theorem for contraction fuzzy mapping. Kang and Cho[13] shows that Heilpern's theorem is a special case of generalized formulations of Caristi-Kirk type fixed point theorem in complete metric space.

It is well-known that the existence of maximal element of a partially ordered set is equivalent to the existence of fixed points of some monotone functions.

In 1955, Tarski [17] proved a fixed point theorem for increasing functions on complete lattices. And Davis [4] proved that the converse of Tarski's Theorem is true. Since then, many authors studied the fixed point theorem of monotone operators.

In 1987, Park [15] showed that Zorn's lemma can be reformulated by various types of fixed point theorems.

In many cases, the iterated sequence $\{f^n(x)\}$ converges to a maximal fixed point of f, where f is a monotone function defined on some topological ordered space X and $x \in X$.

In 1991, Chang and Ma [3] discussed the existence and iterative approximation of coupled fixed points for mixed monotone condensing operators and extended and improved the corresponding results of Guo and Lakshmikantham [9].

In 2004, Chang et. al. [2] proved some fixed point theorems for increasing operators for fuzzy numbers on real line. They applied their theorem to solve fuzzy equation for fuzzy numbers. Their theorems are based in the maximal principle on general ordered sets.

Beg et. al [1] studied the existence of minimal and maximal fixed points for mixed monotone operators on probablistic Banach space. Their proofs are based on the iterative techniques of monotone operators.

In this paper, we will discuss some fixed point theorems for monotone operators in ordered topological space and increasing operators for fuzzy numbers.

In section II, we will introduce some basic definitions of fuzzy sets and fuzzy numbers. We will show that the set of fuzzy numbers can be equipped with a metric using a Hausdorff metric.

We also introduce some definitions and basic concepts about general ordered sets and ordered topological spaces.

In section III, we will prove some fixed point theorems for ordered topological spaces and fuzzy numbers. Also, we will prove a fixed point theorem for condensing map in ordered metric spaces and in the space of fuzzy numbers. And we will prove some coupled fixed point theorems for mixed monotone operator in ordered topological spaces.

II. Preliminaries

In this section, we introduce some definitions and basic results for fuzzy sets and fuzzy numbers. We also consider the basic facts about ordered topological spaces and mixed monotone operator.

Let X be a nonempty set. A function $f : X \to [0,1]$ is called a *fuzzy subset* of X. If f is a fuzzy subset of X, $f(x)$ is called the *grade of membership* of x in f.

Definition 1. Let $x : \mathbb{R} \to [0,1]$ be a fuzzy subset. Then x is called a *fuzzy number* if

(1) the set $x_\alpha = \{\xi \ \mid \ x(\xi) \geq \alpha\}$ is a closed interval $[x_{\alpha_l}, x_{\alpha_r}]$ for each $\alpha > 0$.

(2) $\{\xi \in \mathbb{R} \ \mid \ x(\xi) > 0\}$ is bounded.

The set of all fuzzy numbers is denoted by $\mathbb{R}_L$.

For each fuzzy number x and $\xi \in \mathbb{R}$, we have

$$x(\xi) = \sup\{\alpha > 0 \ \mid \ \xi \in x_\alpha\}.$$

Moreover, the following basic properties are true (see Chang et. al. [2]).

Lemma 1. *For each fuzzy number x, the mapping $\alpha \to x_{\alpha_l}$ and $\alpha \to x_{\alpha_r}$ of $L_0 = \{\alpha \in [0,1] \ \mid \ \alpha > 0\}$ into $\mathbb{R}$ have the following properties.*

(1) $\alpha \to x_{\alpha_l}$ *is increasing,* $\alpha \to x_{\alpha_r}$ *is decreasing and* $x_{\alpha_l} \leq x_{\alpha_r}$*.*

(2) $x_{\alpha_l} = \sup_{0<\beta<\alpha} x_{\beta_l}$ *and* $x_{\alpha_r} = \inf_{0<\beta<\alpha} x_{\beta_r}$ *for all* $\alpha \in L_0$*.*

(3) The infimum $x_{0_l} = \inf_{\alpha>0} x_{\alpha_l}$ *and the supremum* $x_{0_r} = \sup_{\alpha>0} x_{\alpha_r}$ *exist in* $\mathbb{R}$*.*

We will define a metric D on $\mathbb{R}_L$ to show that $(\mathbb{R}_L, D)$ is a metric space.

If (X, d) is a metric space and $A, B \subseteq X$ are bounded subsets of X,

$$d(A, B) = \inf\{\varepsilon > 0 \ \mid \ A \subseteq N(B, \varepsilon), B \subseteq N(A, \varepsilon)\}$$

is called the *Hausdorff metric* between A and B, where

$$N(A, \varepsilon) = \{x \in X \ \mid \ d(a, x) < \varepsilon \ \text{ for some } \ a \in A\}.$$

Define $D : \mathbb{R}_L \times \mathbb{R}_L \to \mathbb{R}_+ \cup \{0\}$ by

$$D(u, v) = \sup_{\alpha \in [0,1]} \{d(u_\alpha, v_\alpha) \ \mid \ u, v \in \mathbb{R}_L\}$$

where $d(A, B)$ is the Hausdorff metric between A and B.

First, we will show that D is a metric on $\mathbb{R}_L$.

Lemma 2. $D : \mathbb{R}_L \times \mathbb{R}_L \to \mathbb{R}_+ \cup \{0\}$ *is a metric.*

Proof. Let $u, v, w \in \mathbb{R}_L$ then,

$$\begin{array}{rcl}
(1)\ \ D(u,v) & = & \sup\limits_{\alpha\in[0,1]} d(u_\alpha, v_\alpha) \ = \ \max\{|u_{\alpha_l} - v_{\alpha_l}|, |u_{\alpha_r} - v_{\alpha_r}|\} \geq 0. \\
(2)\ \ D(u,v) & = & \max\{|u_{\alpha_l} - v_{\alpha_l}|, |u_{\alpha_r} - v_{\alpha_r}|\} \\
 & = & \max\{|v_{\alpha_l} - u_{\alpha_l}|, |v_{\alpha_r} - u_{\alpha_r}|\} \\
 & = & D(v,u) \\
(3)\ \ D(u,v) & = & 0 \text{ if and only if } u_{\alpha_l} = v_{\alpha_l} \text{ and } u_{\alpha_r} = v_{\alpha_r}. \\
\text{That is,} & & D(u,v) = 0 \text{ if and only if } u = v. \\
(4)\ \ D(u,v) & = & \max\{|u_{\alpha_l} - v_{\alpha_l}|, |u_{\alpha_r} - v_{\alpha_r}|\} \\
 & \leq & \max\{|u_{\alpha_l} - w_{\alpha_l}|, |u_{\alpha_r} - w_{\alpha_r}|\} + \max\{|w_{\alpha_l} - v_{\alpha_l}|, |w_{\alpha_r} - v_{\alpha_r}|\} \\
 & = & D(u,w) + D(w,v)
\end{array}$$

By (1), (2), (3) and (4), we have shown that D is a metric. □

Now we will discuss the basic definitions and properties of ordered sets.

Let $\leq$ be an order relation on a nonempty set X. Then $(X, \leq)$ will be called an *ordered set*. For elements $x, y \in X$, $x \geq y$ means $y \leq x$. The terms *upper and lower bound, maximal and minimal element, supremum and infimum etc.* will be used as usual.

Definition 2. Let $(X, \leq)$ be an ordered set. For any $x \in X$, we denote

$$S(x) = \{y \in X \ \mid \ x \leq y\}$$

is called the *super set* of x. And for $u, v \in X$ such that $u \leq v$, the set

$$[u, v] = \{x \in X \ \mid \ u \leq x \leq v\}$$

is called an *order interval*.

Definition 3. Let X be a topological space and $\leq$ an order relation on X. Then $(X, \leq)$ is called an *ordered topological space* if

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$$

for all convergent sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying $x_n \leq y_n$ for all $n \in \mathbb{N}$.

If $(X, \leq)$ is an ordered topological space, $x \in X$ and $\{x_n\}$ is a convergent sequence in $S(x)$, then $x_n \geq x \quad (n = 1, 2, 3, \cdots)$ implies than $\lim_{n\to\infty} x_n \geq x$. Hence $\lim_{n\to\infty} x_n \in S(x)$ and so $S(x)$ is *sequentially closed* that is, every convergent sequence in $S(x)$ converges to a point in $S(x)$.

Obviously, if X is a metric space, then $S(x)$ is closed for all $x \in X$.

Definition 4. Let $(X, \leq)$ be an ordered set. A sequence $\{x_n\}$ in X is said to be *monotone increasing* (*decreasing*, resp.) if

$$x_{n-1} \leq x_n \quad (x_{n-1} \geq x_n, \quad \text{resp.})$$

for all $n = 1, 2, 3, \cdots$.

Definition 5. An ordered topological space $(X, \leq)$ is said to be *sequentially order compact* if every increasing sequence in X has a convergent subsequence.

The following is well-known.

Lemma 3. *Let $(X, \leq)$ be an ordered topological space. If $\{u_n\}$ is an increasing sequence having a convergent subsequence $\{u_{n_k}\}$ and $v = \lim_{k\to\infty} u_{n_k}$, then $v = \sup_n u_n$.*

Lemma 3 shows that any increasing sequence in X having convergent subsequence has a supremum.

Definition 6. Let $(X, \leq)$ be an ordered set and $f : X \to X$ be a function. We say that f is *increasing* if $x \leq y$ implies that $f(x) \leq f(y)$.
For any $x \in X$, the sequence $\{x_n\}$ defined by

$$x_0 = x, \quad x_1 = f(x_0), \quad \cdots, \quad x_n = f(x_{n-1}), \quad n \in \mathbb{N}$$

is called the *iteration* of x by f.

Note that if f is increasing and $x_0 \leq f(x_0)$, then

$$x_0 \leq f(x_0) \leq f(f(x_0)) \leq \cdots$$

and so the iteration of x_0 by f is increasing.

Let X be a nonempty set and $f : X \to X$ be a self function on X. Then a point $x \in X$ is called a *fixed point* of f if $f(x) = x$.

Now consider the following.

Definition 7. Let $(X, \leq)$ be an ordered set and $D = [u, v]$ be an order interval in X. A function $f : D \times D \to X$ is called a *mixed monotone operator* if $f(x, y)$ is increasing in x for each fixed $y \in D$ and decreasing in y for each fixed in $x \in D$. That is,

$$f(x_1, y) \leq f(x_2, y) \ \text{ if } \ x_1 \leq x_2,$$

and

$$f(x, y_1) \geq f(x, y_2) \ \text{ if } \ y_1 \leq y_2.$$

For a mixed monotone operator f, the sequence $\{(x_n, y_n)\}$ in $D \times D$ satisfying

$$x_n = f(x_{n-1}, y_{n-1}), \quad y_n = f(y_{n-1}, x_{n-1}), \quad n = 1, 2, \cdots$$

is called an *iteration* by f.

Definition 8. Let $(X, \leq)$ be an ordered set and $D = [u, v]$ be an order interval in X. Let $f : D \times D \to X$ be a mixed monotone operator.

(1) If $x, y \in D$ with $x \leq y$ satisfies

$$x \leq f(x, y) \ \text{ and } \ f(y, x) \leq y,$$

then (x, y) is called a *coupled lower and upper fixed point of* f.

(2) If $x, y \in D$ with $x \leq y$ can be found such that

$$x = f(x, y) \ \text{ and } \ f(y, x) = y,$$

then (x, y) is called a *coupled fixed point of f*. If a coupled fixed point (x^*, y^*) can be found such that

$$x^* \leq x \quad \text{and} \quad y \leq y^*,$$

for every coupled fixed point (x, y) of f, then (x^*, y^*) is called the *minimal and maximal fixed point* of f.

(3) A point $x^* \in D$ is a *fixed point of f* if

$$f(x^*, x^*) = x^*.$$

We introduce some definitions related to topological spaces and metric spaces.

Definition 9. Let X be a topological space and $A \subseteq X$. We say that A is *relatively compact* if $\overline{A}$ is compact.

We define the measure of noncompactness as follows.

Definition 10. Let X be a metric space and $\mathcal{B}$ the set of all bounded subsets of X. A real valued function $r : \mathcal{B} \to [0, \infty)$ defined by

$$r(S) = \inf\{\varepsilon > 0 \ \mid \ S \ \text{ can be covered by finite number of sets of diam} < \varepsilon\}$$

is called the *measure of noncompactness* of $S \in \mathcal{B}$.

Definition 11. Let X be a metric space and r be the measure of noncompactness. Let $f : X \to X$ be a function. f is said to be *condensing* if it is continuous, bounded and $r(f(S)) < r(S)$ for any bounded set $S \subset X$ with $r(S) > 0$.

Fixed point theorems for condensing operator is originated from Sadovskii[16]. In this paper, we will discuss some fixed points of fuzzy condensing functions.

The following Lemma is well-known (See Kirk[8]).

Lemma 4. *Let X be a metric space and $S \subseteq X$ be a bounded subset. Then the following are equivalent;*
(1) $r(S) = 0$.
(2) For any $\varepsilon > 0$, S can be covered by finite number of set of diam $< \varepsilon$.
(3) $\overline{S}$ is compact.
(4) S is relatively compact.

III. Main results

In this section, we will prove some fixed point theorems for increasing operators in ordered topological space and apply to fuzzy numbers.

To begin with, we prove the following maximal principle on general ordered sets.

Theorem 5. *Let* $(X, \leq)$ *be an ordered topological space,* $u_0, v_0 \in X$ *and* $u_0 \leq v_0$*. Assume that* $B = [u_0, v_0]$ *is a closed set of* X *and every iteration in* B *has a convergent subsequence. Let* $f : B \to B$ *a continuous increasing operator. Suppose that*

$$u_0 \leq f(u_0), \quad f(v_0) \leq v_0.$$

Then f *has a minimal and a maximal fixed point.*

Proof. Let $u_1 = f(u_0), \quad v_1 = f(v_0)$ and

$$u_n = f(u_{n-1}) \quad \text{and} \quad v_n = f(v_{n-1}), \quad n = 1, 2, 3, \cdots .$$

Since f is increasing, it follows from

$$u_0 \leq f(u_0) = u_1, \quad v_1 = f(v_0) \leq v_0$$

that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0$$

holds.

Since the sequence $S = \{u_n\}_{n=0}^{\infty}$ is an iteration, S has a convergent subsequence $\{u_{n_k}\}$.

Let $\lim_{k\to\infty} u_{n_k} = x_* \in X$. By Lemma 3, $x_* = \sup u_n$ and hence $u_n \leq x_*$ for all $n \in \mathbb{N}$.

If $k \in \mathbb{N}$, then $f(u_{n_k}) = u_{n_k+1} \leq x_*$ and the contiuity of f implies that

$$f(x_*) = \lim_{k\to\infty} f(u_{n_k}) \leq x_*$$

Since f is increasing, $u_n \leq x_*$ implies that $f(u_n) \leq f(x_*)$ for all $n \in \mathbb{N}$. Hence $x_* = \lim_{k\to\infty} u_{n_k} = \lim_{n\to\infty} f(u_{n_k-1}) \leq f(x_*)$. Hence $x_* = f(x_*)$ and therefore x_* is a fixed point of f.

Finally, we will prove that x_* is a minimal fixed point of f in B.

Let $x \in B$ and $f(x) = x$. Since f is increasing, it follows from $u_0 \leq x$ that $f(u_0) \leq f(x)$, *i.e.*, $u_1 \leq x$. Assume that $n \geq 1$ and $u_n \leq x$. Then $u_{n+1} = f(u_n) \leq f(x) = x$. By induction, we have $u_n \leq x \quad (n = 1, 2, 3, \cdots)$. So

$$x_* = \sup u_n \leq x.$$

Hence x_* is a minimal fixed point of f. Similarly, we can show that f has a maximal fixed point x^*,

$$x^* = \inf v_n$$

where $\{v_n\}$ is a decreasing sequence defined by $v_n = f(v_{n-1}), \quad n \in \mathbb{N}$. □

From Theorem 5, we can prove a fixed point theorem for continuos increasing operators on ordered topological space as follows.

Corollary 6. *Let $(X, \leq)$ be an ordered topological space, $u_0, v_0 \in X$ and $u_0 \leq v_0$. Let $B \subset [u_0, v_0]$ be a closed subset of X. Let $f : B \to B$ be a continuous increasing operator. Assume that B or $f(B)$ is sequentially order compact.*
Suppose that

$$u_0 \leq f(u_0), \quad f(v_0) \leq v_0.$$

Then f has a minimal fixed point.

Proof. If B or $f(B)$ is sequentially order compact, then every iteration by f has a convergent subsequence. Hence the result follows from Theorem 5. □

Using Theorem 5, we can prove the following fixed point theorem for condensing maps.

Theorem 7. *Let X be an ordered metric space and $B = [u_0, v_0]$ be a closed subset of X. Let $f : B \to B$ a continuous increasing map. Assume that $u_0 \leq f(u_0), \quad f(v_0) \leq v_0$. If f is condensing, then f has a maximal and a minimal fixed point.*

Proof. By Theorem 5, it suffices to show that any iteration by f has a convergent subsequence. Let $S = \{x_n\}_{n=0}^{\infty}$ be an iteration by f, that is $x_1 = f(x_0)$ and $x_n = f(x_{n-1})$ for all $n = 1, 2, \cdots$. Then $S = \{x_0\} \cup f(S)$ implies that $r(S) = r(f(S))$. Since f is condensing, $r(S) = 0$ and so S is relatively compact. Therefore $\{x_n\}$ has a convergent subsequence. □

Corollary 8. *Let the condition of Theorem 7 be satisfied. Suppose that f has only one fixed point x in B. Then for any $x_0 \in B$, the successive interation*

$$x_n = f(x_{n-1}), \quad n = 1, 2, 3, \cdots$$

converges to x.

Proof. Let $B_0 = [x_0, v_0]$, then f maps B_0 into B_0. As in the proof of Theorem 7, the iteration $\{x_n\}$ converges to some fixed point y. Since f has only one fixed point x, $y = x$. Hence $\lim_{n\to\infty} x_n = x$. □

Now we will state and prove some fixed point theorems on the set of fuzzy numbers.

Let $\mathbb{R}_L$ be the set of all fuzzy numbers of $\mathbb{R}$. Define a relation $\leq$ on $\mathbb{R}_L$ by

$$x \leq y \Leftrightarrow x_{\alpha_l} \leq y_{\alpha_l} \text{ and } x_{\alpha_r} \leq y_{\alpha_r} \text{ for all } \alpha \in L_0.$$

Obviously, $\leq$ is an order relation on $\mathbb{R}_L$ and so $(\mathbb{R}_L, \leq)$ is a partially ordered set.

Let D be the metric defined on $\mathbb{R}_L$ in Section II. The following Lemma is due to Chang et. al. [2].

Lemma 9. *Let $x, y, z \in \mathbb{R}_L$. If $x \leq y \leq z$, then $D(z, y) \leq D(z, x)$.*

Using Theorem 5, we can prove the following.

Theorem 10. (Theorem 2.3 in [2]) *Let $B = [u_0, v_0]$ be a closed subset of $\mathbb{R}_L$ and $f : B \to B$ a continuous increasing operator. Assume that f is condensing and $u_0 \leq f(u_0),\ \ f(v_0) \leq v_0$. Then f has a maximal and a minimal fixed point.*

Proof. From Lemma 2, we have known that $(\mathbb{R}_L, D)$ is a metric space. To apply Theorem 5, we will show that $(\mathbb{R}_L, \leq)$ is an ordered metric space.

Let $\{u_n\}, \{v_n\}$ be convergent sequences such that $u_n \leq v_n,\ \ n \in \mathbb{N}$. Let $\lim_{n\to\infty} u_n = u_0$ and $\lim_{n\to\infty} v_n = v_0$.

Since $\lim_{n\to\infty} u_n = u_0$,

$$\lim_{n\to\infty} D(u_n, u_0) = \sup_n \max\{|u_{n_{\alpha_l}} - u_{0_{\alpha_l}}|, |u_{n_{\alpha_r}} - u_{0_{\alpha_r}}|\} = 0$$

implies that $u_{n_{\alpha_l}} \to u_{0_{\alpha_l}}$ and $u_{n_{\alpha_r}} \to u_{0_{\alpha_r}}$ as $n \to \infty$.

Similarly, $v_{n_{\alpha_l}} \to v_{0_{\alpha_l}}$ and $v_{n_{\alpha_r}} \to v_{0_{\alpha_r}}$ as $n \to \infty$.

Since $v_{n_{\alpha_l}} \geq u_{n_{\alpha_l}}$ and $v_{n_{\alpha_r}} \geq u_{n_{\alpha_r}}$ for all $n \in \mathbb{N}$, and we have

$$v_{0_{\alpha_l}} \geq u_{0_{\alpha_l}} \quad \text{and} \quad v_{0_{\alpha_r}} \geq u_{0_{\alpha_r}}.$$

This show that $u_0 \leq v_0$ and hence $(\mathbb{R}_L, \leq)$ is an ordered metric space.

Hence the result follows from Theorem 5. □

Remark. In Theorem 10, the iteration $\{u_n\}$ given by

$$f(u_0) = u_1, \quad \text{and} \quad f(u_{n-1}) = u_n, \quad n = 1, 2, 3, \cdots$$

converges to a minimal fixed point x_*. Indeed, if $m, k \in \mathbb{N}$ and $m > n_k$, $u_{n_k} \leq u_m \leq x_*$ implies that $D(x_*, u_m) \leq D(x_*, u_{n_k}) \to 0$ as $n_k \to \infty$, by Lemma 2.

Theorem 10 can be used to prove the following existence theorem of fuzzy equation;

$$Ex^2 + Fx + G = x,$$

where E, F, G, and x are fuzzy numbers (see Chang et. al. [2]).

The following is a fixed point theorem for mixed monotone operators.

Theorem 11. *Let X be an ordered topological space and $u_0, v_0 \in X$, $u_0 \leq v_0$. Let B a closed subset of $[u_0, v_0]$ such that $u_0, v_0 \in \mathbb{R}_L$. Let $f : B \times B \to B$ be a mixed monotone operator such that*

$$u_0 \leq f(u_0, v_0), \quad f(v_0, u_0) \leq v_0.$$

Suppose that f is continuous and every iteration by f in $B \times B$ has a convergent subsequence. Then f has a coupled fixed point $(x_*, y^*) \in B \times B$, which is minimal and maximal in the sense that $x_* \leq x \leq y^*$ and $x_* \leq y \leq y^*$ for any coupled fixed point $(x, y) \in B \times B$ of f. Moreover, we have

$$x_* = \lim_{n \to \infty} u_n, \quad y^* = \lim_{n \to \infty} v_n,$$

where $u_n = f(u_{n-1}, v_{n-1})$ and $v_n = f(v_{n-1}, u_{n-1})$, $n = 1, 2, 3, \cdots$, which satisfy

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Proof. First, we define an order relation $\preceq$ on $B \times B$ by

$$(x, y) \preceq (z, w) \Leftrightarrow x \leq z \text{ and } w \leq y.$$

Then it is easy to show that $B \times B = [(u_0, v_0), (v_0, u_0)]$ is an order interval in $B \times B$.

Define $F : B \times B \to B \times B$ by

$$F(x, y) = (f(x, y), f(y, x)).$$

If $(x, y) \preceq (z, w)$, then

$$f(x, y) \leq f(z, y) \leq f(z, w) \quad \text{and} \quad f(y, x) \geq f(w, x) \geq f(w, z)$$

implies that

$$F(x, y) = (f(x, y), f(y, x)) \preceq (f(z, w), f(w, z)) = F(z, w).$$

Hence F is increasing.

Assume that $\{(x_n, y_n)\}$ is an iteration by F. Then

$$(x_n, y_n) = F(x_{n-1}, y_{n-1}) = (f(x_{n-1}, y_{n-1}), f(y_{n-1}, x_{n-1}))$$

implies that

$$x_n = f(x_{n-1}, y_{n-1}), \quad y_n = f(y_{n-1}, x_{n-1}).$$

Hence $\{(x_n, y_n)\}$ is a iteration by f. By assumption, it has a convergent subsequence. By Theorem 5, F ha a maximal fixed point (x_0, y_0) in $B \times B$.

Obviously, (x_0, y_0) is a coupled fixed point of f, which is minimal and maximal in the sense given above. $\square$

We say that f is completely continuous if f is continuous and $\overline{f(S)}$ is compact for any bounded subset S of $B \times B$ (see Chang et. al[2]). Note that if f is completely continuous then every iteration by f has a convergent subsequence. Hence we have the following for fuzzy numbers.

Corollary 12. (Chang. et. al[2]) *Let* $u_0, v_0 \in \mathbb{R}_L, \quad u_0 \leq v_0$ *and* B *a closed subset of* $[u_0, v_0]$*. Let* $f : B \times B \to B$ *be a mixed monotone operator such that*

$$u_0 \leq f(u_0, v_0), \quad f(v_0, u_0) \leq v_0.$$

Suppose that f *is complete continuous, then* f *has a coupled fixed point* $(x_*, y^*) \in B \times B$*, which is minimal and maximal in the sense that* $x_* \leq x \leq y^*$ *and* $x_* \leq y \leq y^*$ *for any coupled fixed point* $(x, y) \in B \times B$ *of* f*. Moreover, we have*

$$x_* = \lim_{n \to \infty} u_n, \quad y^* = \lim_{n \to \infty} v_n,$$

where $u_n = f(u_{n-1}, v_{n-1})$ *and* $v_n = f(v_{n-1}, u_{n-1}), \quad n = 1, 2, 3, \cdots,$ *which satisfy*

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

References

1. I. Beg, A. Latif, R. Ali and A. Azam, *Coupled fixed point of mixed monotone operators on probablistic banach space*, Archivum Mathematicum(Brno)., 37 (2001), 1-8.

2. Chang et. al., *Fixed-point theorems in fuzzy real line*, Computers and Mathematics with Application 47 (2004), 845-851.

3. S. Chang, Y. H. Ma, *Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming*, Jour. Math. Anal. Appl. 160 (1991), 468-479.

4. A. C. Davis, *Characterization of complete lattice*, Paciffic J. Math., 5 (1995), 311-319.

5. James DugumDj; and andrzej grannas, *Fixed Point Theory Volume 1*, (1982).

6. Nelson Dunfor D and Jacob T. Schwartz,*Linear Operators Part I*, (1957).

7. A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems 64 (1994), 395-399.

8. K. Goebel and W. A. Kirk,*Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, (1990).

9. D. Guo and V. Lakshmikantham, *Coupled fixed point of nonlinear operators with applications*, Nonlinear Analysis. TMA., 11 (1987), 623-632.

10. S. Heikkilila, *On operator and integral equations with discontinuous righthand side*, Jour. Math. Anal, Appl., 140 (1989), 200-217.

11. S. Heilpern, *Fuzzy mappings and fixed point theorem*, J. Math. Anal. Appl. 83 (1981), 566-569.

12. O. Kalera and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems 12 (1984), 215-229.

13. B. G. Kang and S. J. Cho, *An application of ordering principle to fixed points of fuzzy mappings*, Nonlinear Analysis Forum 1 (1995), 43-52.

14. I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika 11 (1975), 336-344.

15. S. Park, *Equivalent formulation of Zorn's lemma and other maximum principle*, J. Korea Soc.Math.Ed., 25 (1987), 19-27.

16. B. N. Sadovskii, *Limit compact condensing operators*, Russian Math Surveys 27, (1972), 85-155.

17. A. Tarski, *A lattice theoretic fixpoint theorems and its applications*, Paciffic J. Math., 5 (1955), 285-310.

18. L.A. Zadeh, *Fuzzy Sets*, Inform. and Control 8 (1965), 338-353.

Abstract

Fixed Points of Increasing Operators on Fuzzy Numbers

Since Zadeh introduced the concept of fuzzy set theory, many authors showed its wide applications to various fields, and proved a lot of properties of structures and functions on fuzzy sets.

Recently, it is shown that the set of fuzzy sets can be made into metric spaces by defining some metric, and the fixed point theory on fuzzy metric spaces are studied widely.

In 2004, Chang et. al. proved some fixed point theorems for increasing operators for fuzzy numbers on real line. They applied their theorem to solve fuzzy equation for fuzzy numbers. Their theorems are based in the maximal principle on general ordered sets.

In this paper, we showed fixed point theorems for increasing operators on general ordered sets and some fixed point theorems on fuzzy numbers. Also we proved a fixed point theorem for condensing map in ordered metric space.

Moreover, we proved a coupled fixed point theorem for mixed point monotone operators in ordered topological space.