姜秉介 教授指導

碩士學位 請求論文

Fixed Points of Increasing Multivalued Maps

2006

誠信女子大學校 教育大學院

教育學科 數學教育專攻

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認 准 書

李 美 珍의 碩士學位 論文을 認准함

審査委員 ________________ 印

審査委員 ________________ 印

審査委員 ________________ 印

2006 年 6 月

誠信女子大學校 敎育大學院

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논 문 개 요

순서집합(順序集合)에서의 부동점 이론(不動點 理論)은 Zermelo, Tarski 등에 의하여 연구되었다. 1908년에 Zermelo는 X가 모든 전순서 부분집합의 상계를 가지는 순서집합일 때, $f : X \to X$ 가 모든 $x \in X$에 대하여 $x \leq f(x)$를 만족시키면 f는 부동점을 가진다는 사실을 증명하였고, 1955년에 Tarski는 완비속(完備束 - complete lattice)에서의 증가함수 부동점 정리를 증명하였다.

Zermelo와 Tarski의 부동점 이론은 대수적이나 위상적인 조건이 없이 순수하게 일반 순서집합에서의 부동점을 다루었다는 점에서 의미가 있다. 함수의 부동점의 존재는 근본적으로 극대원(極大元)의 존재와 주어진 순서집합의 완비성(完備性)과 관계가 있다.

Smithson [11]은 1971년에 다가함수(多價函數 - multivalued function)의 증가, 감소를 정의하고 순서집합에서 다가함수의 부동점 정리를 증명한 바 있다. 그는 Zermelo와 Tarski의 결과를 다가함수에 대하여 일반화했다.

이 논문에서는 Carl과 Heikkila [2]가 증명한 다가함수의 부동점 정리의 다른 증명을 보인다.

또 순서가 주어진 제 2가산집합(ordered second countable space)에서는 Carl과 Heikkila의 조건을 가정하지 않고도 부동점 정리가 성립함을 보이고, 몇 가지 다른 부동점 정리도 증명한다.

I. Introduction

Let X be a partially ordered set with an order relation $\leq$ and $f : X \to X$ a self-function. We say that f is *increasing* if $x \leq y$ implies that $f(x) \leq f(y)$ for all $x, y \in X$. Also a multivalued function $F : X \to 2^X \setminus \{\emptyset\}$ is said to be *increasing* if $x \leq y$ and $v \in F(x)$ implies the existence of a $w \in F(y)$ such that $v \leq w$.

Fixed Point theorems for functions in ordered sets are originated from the following theorem of Zermelo (See Dunford and Schwartz [4] and Kirk [5]).

Theorem (Zermelo). *Let X be a partially ordered set and assume that every chain in X has a supremum. Then every selfmap $f : X \to X$ satisfying*

$$x \leq f(x), \quad x \in X$$

has a fixed point.

It is well-known that the above fixed point theorem is equivalent to Zorn's Lemma. Indeed, Park [9] showed that Zorn's Lemma can be reformulated to some fixed point theorems for single valued or multivalued functions satisfying Zermelo type conditions.

Obviously, any maximal element in X is a fixed point of self-function of f in the Zermelo's Theorem. Therefore we can see that fixed point theorems in ordered sets are deeply related to maximal principles.

In 1955, Tarski [12] proved that every increasing function defined on a complete lattice has a fixed point. Since then, many authors gave interests to fixed points for increasing operators on ordered sets. In Tarski's Theorem, the completeness of the given lattice has a crusial role.

In 1971, Smithson [11] obtained a fixed point theorem for multivalued functions in partially ordered sets. He extended the notion of monotonicity of functions to multivalued functions and generalized Zermelo's and Tarski's results.

In 2004, Carl and Heikkila [2] proved some fixed point theorems for multivalued functions on some ordered topological spaces and applied them to prove some differential equations. They used the following assumption to prove the existence of fixed point on an ordered topological vector space X;

(C) Each well-ordered chain C of X whose increasing sequences converge contains an increasing sequence which converges to $\sup C$.

Note that if (C) holds, then every convergent increasing sequence in X has a supremum.

In this paper, we will prove some fixed point theorems for multivalued functions on ordered topological spaces. And we will show that Carl-Heikkila type fixed point theorem on second countable spaces can be proved without assuming the condition (C).

In Section II, we will discuss some basic definitions and fundamental facts about ordered sets.

In Section III, as main results, we will state and prove some fixed point theorems for multivalued functions on ordered topological spaces.

First, we will find another proof of Carl and Heikkila's fixed point theorem [2]. And we will show and prove some fixed point theorems on ordered second countable spaces without assuming (C). Also we will establish some other fixed point theorems for multivalued increasing operators on ordered second countable topological spaces.

II. Preliminaries

A nonempty set X with an order relation $\leq$ is called an *ordered set* and is denoted by $(X, \leq)$. The terms *upper bound, lower bound, supremum, infimum*, etc. will be used as usual.

Let $(X, \leq)$ be an ordered set. For any $x \in X$, we denote

$$S(x) = \{y \in X \mid x \leq y\} \text{ and } L(x) = \{y \in X \mid y \leq x\}.$$

If $u, v \in X$ and $u \leq v$, we denote $[u, v] = \{x \in X \mid u \leq x \leq v\}$ and $[u, v]$ is called an *order interval* in X.

In the following, we define the monotonicity of single valued and multi-valued functions in general ordered sets.

Definition 1. Let $(X, \leq)$, $(Y, \preceq)$ be ordered sets and $f : X \to Y$ a function, then f is said to be *increasing* if

$$x \leq y \Longrightarrow f(x) \preceq f(y), \quad x, y \in X.$$

A multivalued function $F : X \to 2^X \setminus \{\emptyset\}$ is said to be *increasing* if $u, v \in X$, $u \leq v$ and $x \in F(u)$ imply an existence of $y \in F(v)$ such that $x \leq y$.

Definition 2. Let $(X, \leq)$, $(Y, \preceq)$ be ordered sets. We say that X and Y are isomorphic if there is a bijective function $f : X \to Y$ such that

$$x \leq y \Longleftrightarrow f(x) \preceq f(y), \quad x, y \in X.$$

As usual, if every nonempty subset of an ordered set X has the first element, then X is called a *well-ordered set*.

Let $(X, \leq)$ be a well-ordered set. For any $x \in X$, if the set

$$\{y \in X \mid x \neq y \ \text{ and } \ x \leq y\}$$

is nonempty, then the first element of this set is called the *immediate successor* of x and is denoted by $x+1$. If x does not have any immediate successor, then x is called the *last element of* X.

Let B be a well-ordered set, and $A \subset B$. We say that A is an *initial segment* of B if there is a $b \in B$ such that

$$A = \{x \in B \mid x < b\}.$$

It is well-known that for any two well-ordered sets A and B, one of them is isomorphic to an initial segment of the other unless they are isomorphic (See Pinter [10]).

So every well-ordered set X can be indexed by a well-ordered set Λ by $X = \{x_\alpha \mid \alpha \in \Lambda\}$ which satisfies

$$x_\alpha \leq x_\beta \Longleftrightarrow \alpha \leq \beta, \quad \alpha, \beta \in \Lambda.$$

Let $(X, \leq)$ be an ordered set and consider the following conditions;

(A) Every nonempty chain in X has a supremum.

(B) Every nonempty chain in X has an upper bound.

(C) Every nonempty well-ordered subset of X has a supremum.

(D) Every nonempty well-ordered subset of X has an upper bound.

Zorn's lemma asserts that if X satisfies (B), then X has a maximal element. It is well-known that (B) can be replaced by any one of (A), (C) or (D) (See Bourbaki [1]). Moreover, Park [9] showed that Zorn's lemma can be reformulated to some fixed point theorems as follows;

Theorem 1. (Park, [9]) *Let $(X, \leq)$ be an ordered set. Suppose that one of (A),(B),(C) or (D) holds. Then the following equivalent statement holds;*

(a) X has a maximal element.

(b) Every selffunction $f : X \to X$ satisfying

$$x \leq f(x), \quad x \in X$$

has a fixed point.

(c) Every multivalued function $F : X \to 2^X \setminus \{\emptyset\}$ satisfying

$$\forall x \in X, \quad \exists y \in F(x), \quad x \leq y$$

has a fixed point.

(d) Every multivalued function $F : X \to 2^X \setminus \{\emptyset\}$ satisfying

$$\forall x \in X, \quad \exists y \in F(x), \quad x \leq y$$

has a point $x_0 \in X$ *such that* $F(x_0) = \{x_0\}$.

Throughout this paper, all topological spaces are assumed to be Hausdorff. Let X be a topological space and $C \subseteq X$. We say that C is *sequentially closed* if every convergent sequence in C has the limit in C.

Now consider the following definitions for ordered topological space.

Definition 3. Let X be a topological space with an order relation $\leq$. Then $(X, \leq)$ will be called an *ordered topological space* if

$$x_n \leq y_n \Rightarrow \lim x_n \leq \lim y_n$$

for all convergent sequence $\{x_n\}$ and $\{y_n\}$.

If $\{x_n\}$ is a convergent sequence in an ordered topological space X and y is an upper bound of $\{x_n\}$, then $x_n \leq y$ for all $n \in \mathbb{N}$ implies that $\lim_{n\to\infty} x_n \leq y$.

Definition 4. Let $(X, \leq)$ be an ordered topological space. If $S(x)$ and $L(x)$ are closed for all $x \in X$, then $\leq$ is called a *closed order*.

Remark. If $(X, \leq)$ is an ordered topological space and $\{x_n\}$ is a convergent sequence in $S(x)$, then $x \leq x_n \quad (n \in \mathbb{N})$ implies that $x \leq \lim_{n\to\infty} x_n$. Hence $\lim_{n\to\infty} x_n \in S(x)$.

This shows that $S(x)$ is sequentially closed for each $x \in X$. Similarly, we can show that $L(x)$ is sequentially closed, too. So if X is an ordered metric space, then $\leq$ is always closed.

Moreover, the following is true;

Lemma 2. *Let X be an ordered topological space with an order relation $\leq$. Then for any increasing sequence $\{x_n\}$ in X, $\lim_{k\to\infty} x_{n_k}$ is the supremum of $\{x_n\}$ for all convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$.*

Proof. Let $x_0 = \lim_{k\to\infty} x_{n_k}$. For any $k_0 \in \mathbb{N}$, $\{x_{n_k}\}_{k\geq k_0} \subset S(x_{n_{k_0}})$ implies that $x_0 = \lim_{k\to\infty} x_{n_k} \geq x_{n_{k_0}}$. Since k_0 was arbitary, x_0 is an upper bound for the subsequence $\{x_{n_k}\}$.

Assume that y is an upper bound for $\{x_{n_k}\}$. Then $x_{n_k} \leq y$ implies that $x_0 = \lim_{k\to\infty} x_{n_k} \leq y$. Therefore $x_0 = \sup\{x_{n_k}\}$.

For any $k \in \mathbb{N}$, $k \leq n_k$ implies that $x_k \leq x_{n_k} \leq x_0$. Hence x_0 is an upper bound for $\{x_n\}$. If y is an upper bound for $\{x_n\}$, then it is also an upper bound for $\{x_{n_k}\}$. Hence $x_0 \leq y$. Therefore, $x_0 = \sup x_n$.

Definition 5. Let $(X, \leq)$ be an ordered topological space. We say that X is *sequentially order compact* if every increasing sequence in X has a convergent subsequence.

From Lemma 2, we can show that every convergent increasing sequence in X has a supremum.

Corollary. *Let $(X, \leq)$ be an ordered topological space. If X is sequentially order compact, then every increasing sequence in X has a supremum.*

Proof. Let $\{x_n\}$ be an increasing sequence. Since X is sequentially order compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. By Lemma 2, $\lim_{k\to\infty} x_{n_k} = \sup x_n$. Hence $\{x_n\}$ has a supremum.

III. Main results

We will prove some fixed point theorems for multi-valued increasing functions on some ordered topological spaces.

To begin with, we will prove;

Lemma 3. *Let $(X, \leq)$ be an ordered set and $F : X \to 2^X \setminus \{\emptyset\}$ an increasing multivalued function. Assume that*

(3.1) $P_0 = \{x \in X |$ there exists a $y \in F(x)$ such that $x \leq y\}$ is nonempty.

Then there is a maximal well-ordered set C which satisfies the property;

(3.2) $x + 1 \in F(x)$ for all $x \in C$ having the immediate successor.

Proof. Let $\mathcal{A} = \{C \subseteq P_0 \mid C$ is well-ordered, $x + 1 \in F(x)$ for all $x \in C$ having the immediate successor$\}$.

Let $u_0 \in P_0$ and $C_0 = \{u_0\}$. Then $C_0 \in \mathcal{A}$ and so $\mathcal{A} \neq \emptyset$.

Define relation $\preceq$ on $\mathcal{A}$ by

$$C \preceq D \Longleftrightarrow C = D \text{ or } C \text{ is an initial segment of } D.$$

Then it is well-known that $\preceq$ is an order relation on $\mathcal{A}$. We will show that $\mathcal{A}$ has a maximal element.

Suppose that Σ is a nonempty well-ordered subset of $\mathcal{A}$.

We will show that $C = \bigcup \Sigma$ is an upper bound of Σ. Since Σ is well-ordered, Σ can be indexed by $\Sigma = \{C_\alpha \mid \alpha \in \Lambda\}$ where Λ is a well-ordered set such that

$$\alpha \leq \beta \iff C_\alpha \preceq C_\beta \text{ for all } \alpha, \beta \in \Lambda.$$

If D is a nonempty subset of C, the set $\Lambda_0 = \{ \alpha \in \Lambda \mid D \bigcap C_\alpha \neq \varnothing\}$ is a nonempty subset of Λ. And so it has the first element α_0.

Since $D \bigcap C_{\alpha_0}$ is a nonempty subset of a well-ordered set C_{α_0}, it has the first element u_0. For all $u \in D$, there is a $\beta \in \Lambda$ such that $u \in C_\beta$. Then $\beta \in \Lambda_0$ and so $\alpha_0 \leq \beta$. But then $C_{\alpha_0} = C_\beta$ or C_{α_0} is an initial segment of C_β. Since u_0 is the first element of C_{α_0}, $u_0 \leq u$. Hence u_0 is the first element of D.

This shows that C is a well-ordered subset of X. Moreover,

$$C = \bigcup \Sigma = \bigcup \{C_\alpha \mid \alpha \in \Lambda\}$$

implies that $C_\alpha \preceq C, \quad \alpha \in \Lambda$.

Hence C is an upper bound of Σ. By Zorn's lemma, $\mathcal{A}$ has a maximal element C_0.

Let $(X, \leq)$ be an ordered topological space. Carl and Heikkila [2] used the following condition (C) to establish some fixed point theorems for multivalued operators in ordered Banach spaces;

(C) Each well-ordered chain C of X whose increasing sequences converge contains an increasing sequence which converges to $\sup C$.

Using Lemma 3, we can obtain another proof of the following fixed point theorem;

Theorem 4. (Lemma 2.2 in [2]) *Let $(X, \leq)$ be an ordered topological space satisfying (C) and $F : X \to 2^X \setminus \{\emptyset\}$ be a multivalued function. Assume that (3.1) holds and*

(4.1) If $x_n \leq y_n \in F(x_n), n \in \mathbb{N}$ and if $\{y_n\}$ is increasing, then $\{y_n\}$ has a limit in P_0.

Then F has a maximal fixed point which is also a maximal element of P_0.

Proof. By Lemma 3, there is a maximal well-ordered subset C_0 of P_0 satisfying (3.2). Let

$$D_0 = \{y \in P_0 \mid y \text{ is the last element of } C_0 \text{ or } y = x+1 \text{ for some } x \in C_0\}.$$

Then obviously, D_0 is a well-ordered subset of P_0.

Assume that $\{y_n\}$ is an increasing sequence in D_0. Then $y_n = x_n + 1$ for some $x_n \in C_0$ or y_n is the last element of C_0, $n \in \mathbb{N}$.

Since $x_n \leq y_n = x_n + 1 \in F(x_n)$ and $y_n \leq y_{n+1}$ for all $n \in \mathbb{N}, \{y_n\}$ has a limit in P_0, by (4.1). Since X satisfies (C), D_0 has an increasing sequence $\{y_n\}$ which converges to $v_0 = \sup D_0$.

If $x \in C_0$ and x is the last element of C_0, then $x \in D_0$ and so $x \leq v_0$.

If $x \in C_0$ and x has the immediate successor $x+1$, then $x+1 \in D_0$ and $x < x+1 \leq v_0$.

Therefore v_0 is an upper bound of C_0 and hence $C_0 \bigcup \{v_0\}$ satisfies (3.2).

By maximality of C_0, we conclude that $v_0 \in C_0$. Obviously, v_0 is a maximal element of P_0 and $v_0 \in F(v_0)$.

If $(X, \leq)$ is an ordered topological space with some separability assumption, we can prove a fixed point theorem without assuming (C).

Theorem 5. *Let $(X, \leq)$ be an ordered topological space. Assume that*

(5.1) every nonempty well-ordered subset of X has a countable dense subset,

(5.2) X is sequentially order compact.

Let $F : P \to 2^P \setminus \{\emptyset\}$ satisfy (3.1), (4.1). Then F has a maximal fixed point.

Proof. By Lemma 3, there is a maximal well-ordered set C_0 of P_0 satisfying (3.2). By assumption, C_0 has a countable dense subset $\{u_n\}_{n=1}^{\infty}$. After rearranging, we may assume that $\{u_n\}$ is increasing, by (5.2), $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$. Let $\lim_{k\to\infty} u_{n_k} = v$. Then $v = \sup u_n$.

Let $x \in C_0$. Since $\{u_n\}$ is dense in C_0, $x = \lim_{k\to\infty} u_{n_k}$ for some subsequence $\{u_{n_k}\}$ of u_n. Since $u_{n_k} \leq v$ for all $k \in \mathbb{N}, x \leq v$. Hence v is an upper bound of C_0. Since $v = \sup u_n$, we see that $v = \sup C_0$. As in the proof of Theorem 4, $v_0 \in C_0$ and v_0 is a maximal element of P_0. Hence $v_0 \in F(v_0)$.

The following is a basic maximal principle in Kang [7].

Theorem 6. *Let $(X, \leq)$ be an ordered second countable topological space. Assume that X is sequentially order compact. Then X has a maximal element.*

Using Theorem 6, we can prove the following;

Theorem 7. *Let $(X, \leq)$ be an ordered second countable topological space. Let $F : X \to 2^X \setminus \{\emptyset\}$ be an increasing operator. Assume that (3.1) holds and P_0 is sequentially order compact. Then F has a fixed point.*

Proof. If $x \in P_0$, then there exists a $y \in F(x)$ such that $x \leq y$. Since F is increasing, there exists a $z \in F(y)$ such that $y \leq z$. This shows that $y \in P_0$ and hence the multivalued function defined by $G(x) = F(x) \bigcap P_0$ maps P_0 into $2^{P_0} \setminus \{\emptyset\}$.
By Theorem 6, P_0 has a maximal element x_0. But then there exists a $y \in G(x_0) = F(x_0) \bigcap P_0$ such that $x_0 \leq y$. By the maximality of x_0, we have $y = x_0 \in F(x_0)$ and so x_0 is a fixed point of F.

Obviously, if $f : X \to X$ is a single valued function, we have the following;

Corollary. *Let $(X, \leq)$ be an ordered second countable topological space and $f : X \to X$ be increasing. If the set $P_0 = \{x \in X \mid x \leq f(x)\}$ is nonempty and sequentially order compact, then f has a fixed point.*

Note that the above Corollary is equivalent to Theorem 6 in Kang[7].

Theorem 8. *Let $(X, \leq)$ be an ordered second countable topological space and P be a sequentially order compact subset of X.*

Let $F : P \to 2^P \setminus \{\emptyset\}$ be an increaing operator. Assume that (3.1) holds and

(8.1) For all $u \in P$, if $v_1, v_2 \in F(u)$, there exists a $v \in F(u)$ such that $v_1 \leq v$ and $v_2 \leq v$.

(8.2) For each $u \in P$, the value $F(u)$ of F is sequentially compact.

Then F has a maximal fixed point.

Proof. Note that the set $P_0 = \{u \in P \mid u \leq v \quad \text{for} \quad \text{some} \quad v \in F(u)\}$ is nonempty and second countable.

Let $\{u_n\}$ be an increasing sequence in P_0. Since F is increasing, we can choose an increasing sequence $\{v_n\}$ such that $v_n \in F(u_n), u_n \leq v_n$. Since P is sequentially order compact, $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$. Let $v = \lim\limits_{k \to \infty} u_{n_k}$. Then $v = \sup u_n$ by lemma 2, and hence $u_n \leq v$ for all $n \in \mathbb{N}$.

Since F is increasing, there exist $w_n \in F(v)$ such that $v_n \leq w_n$. By (8.1), we may assume that w_n is increasing. Since $F(v)$ is sequentially compact, $\{w_n\}$ has a convergent subsequence $\{w_{n_k}\}$. Let $\lim_{k\to\infty} w_{n_k} = w_0 \in F(v)$. By lemma 2, $w_0 = \lim_{k\to\infty} w_{n_k} = \sup w_n$. But $u_n \leq v_n \leq w_n$ for all $n = 1, 2, \cdots$ implies that $w_0 \geq \sup u_n = v$. So $v \in P_0$ and hence P_0 is sequentially order compact. By Theorem 7, F has a fixed point.

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Abstract

The fixed point theory on ordered sets have been studied by Zermelo, Tarski and others. In 1908, Zermelo proved that if X is a partially ordered set any of whose nonempty chain has an upper bound, then every function $f : X \to X$ satisfying $x \leq f(x)$, $x \in X$ has a fixed point. And in 1955, Tarski proved that every increasing function defined on a complete lattice has a fixed point.

The Zermelo and Tarski 's fixed point theory is meaningful because they are only related to ordering structure and independent of algebraic and topological structures. The existence of the fixed points of functions is basically related to the subsistence of maximal element and the completeness of given ordered sets.

In 1971, Smithson defined the monotonicity of multivalued functions and proved a fixed point theorem for multivalued function in ordered sets. He generalized the Zermelo and Tarski's result.

In this paper, we show another proof of the fixed point for multivalued functions given by Carl and Heikkila [2].

And we prove some other fixed point theorems for multivalued functions on ordered second countable spaces.