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碩士學位 請求論文

Fixed Points of Fuzzy Contractive Maps

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논 문 개 요

1969년 Nadler는 완비거리공간에서 Banach의 축소사상원리(contraction principle)를 다가함수(multivalued function)의 부동점 정리로 일반화 하였다.

또한 1976년 Caristi는 완비거리공간에서 연속이 아닌 함수에 대한 한 부동점 정리를 증명하였는데, 이 정리는 Banach의 축소사상원리를 일반화 한 것이다.

한편, 퍼지 집합의 이론은 1965년에 Zadeh에 의하여 소개된 이래로 많은 학자들에 의하여 그 풍부한 응용성이 입증되었다.

2002년 Frigon 과 O'Regan은 퍼지함수의 Banach형 부동점정리를 증명하였다. 그런데 퍼지함수의 부동점정리로 일반 거리공간에서의 다가함수(multivalued function)의 부동점정리로부터 증명할 수 있다.

이 논문에서는 Caristi의 부동점 정리를 이용하여 Nadler의 정리를 증명하였다. 또, Caristi의 정리를 이용하여 퍼지함수에 관한 Frigon과 O'Regan의 부동점정리를 증명하였다.

또한, 퍼지사상의 부동점정리를 완비거리공간의 일반적인 multivalued 사상의 부동점 정리로부터 증명할 수 있음을 보였다.

목 차

논문개요

I Introduction

Let (X, d) be a metric space and $f : X \to X$ a selfmap. We say that f is a contraction mapping if there is a real number $k, \quad 0 < k < 1$ such that

$$d(f(x), f(y)) \leq kd(x, y), \quad x, y \in X.$$

In 1922, Banach proved the following theorem, which is called the Banach contraction principle;

Theorem I.1 (Banach). *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction mapping. Then f has a fixed point. That is, there is an $x \in X$ such that $f(x) = x$.*

Banach's contraction principle has wide applications in analysis and so there have been a lot of generalizations and reformalations.

In 1969, Nadler[10] generalized the Banach contraction principle for multivalued maps as follows;

Theorem I.2 (Nadler). *Let (X, d) be a complete metric space and $CB(X)$ be the set of all closed and bounded subsets of X. Let $F : X \to 2^X \setminus \{\emptyset\}$ be* *a multivalued map. Suppose that there exists a* $k, 0 < k < 1$ *such that*

$$H(F(x), F(y)) \leq kd(x, y), \quad x, y \in X$$

where H *is the Hausdorff distance on* $CB(X)$*. Then* F *has a fixed point, that is, there exists an* $x \in X$ *such that* $x \in F(x)$*.*

Also in 1976, Caristi [3] proved a fixed point theorem in complete metric spaces without assumming the continuity of given function. Caristi's Theorem generalizes the Banach contraction principle.

Theorem I.3 (Carsti-Kirk theorem). *Let* (X, d) *be a complete metric space and* $\phi : X \to \mathbb{R}$ *be a lower semi continuous function which is bounded from below. Suppose that* $f : M \to M$ *is an arbitrary mapping which satisfies*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)), \quad x \in X.$$

Then f *has a fixed point.*

On the other hand, the fuzzy set theory was introduced by Zadeh [13] in 1965 and emerged an interesting branch of pure and applied sciences. Since then, a lot of structures on fuzzy sets are obtained and many authors have developed the fuzzy sets and their applications. Of course, a lot of fixed point theorems for fuzzy mappings have been studied.

Heilpern[7] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mapping which is the fuzzy analogue of the Nadler's fixed point theorem[9] for multivalued mappings.

Bose and Sahani[2], Vijayaraju and Maruai[12] extended Heilpern's resutts for generalized fuzzy contraction mappings and for fuzzy multivalued mappings.

Kang and Cho[8] shows that Heilpern's theorem is a special case of generalized formulations of Caristi-Kirk type fixed point theorem in complete metric space.

In 2002, Frigon and O'Regan[5] proved a Banach type fixed point theorem for fuzzy mappings.

In this paper, we will prove some fixed point theorems for fuzzy contract on mappings. We will show that fuzzy fixed point results can be deduced from the fixed point theory of multivalued mappings with closed values.

We will show that Nadler's fixed point theorem for multi-valued contraction mappings can be deduced from Caristi's theorem.

Also We will prove Frigon and O'Regan's Theorem using Caristi's theorem.

Moreover, we will show that some fixed point theorems for fuzzy mappings can be proved from the fixed point theorems for general multivalued mappings on complete metric spaces.

Our methods are based on some maximal principle on ordered sets.

II Preliminaries

For any nonempty set X, a function $f : X \to X$ is called a *selfmap* of X. Let $f : X \to X$ be a selfmap of X. A point $x \in X$ satisfying $f(x) = x$ is called a *fixed point* of f.

If $F : X \to 2^X \backslash \{\emptyset\}$ is a function, then we say that F is a multivalued function, where 2^X denotes the power set of X. Also, a point $x \in X$ is called a *fixed point* of F if $x \in F(x)$.

Definition II.1. Let (X, d) be a complete metric space and $f : X \to X$ be a selfmap. We say that f is a *contraction* mapping if there is a real constant $k \in [0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$.

The following is the well-known Banach contraction principle;

Theorem II.1 (Banach contraction principle). *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point ξ, and $\{f^n(x)\}$ converges to ξ for each $x \in X$.*

Note that every contraction mapping on a metric space is continuous.

In 1976, Caristi[3] generalized Banach contraction principle without assuming the continuity. His condition is related to lower semicontinuous real function on metric space.

The following is the definition of lower semicontinuous function.

Definition II.2. Let (X, d) be a metric space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ a function. Then ϕ is said to be *lower semicontinuous* at $x_0 \in X$ if

$$\phi(x_0) \leq \liminf_{x \to x_0} \phi(x).$$

We say that ϕ is *lower semicontinuous* on X if ϕ is lower semicontinuous at every point in X.

It is well-known that lower semicontinuous functions have the following basic properties.

Definition II.3. Let (X, d) be a metric space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ a function. Then the followings are equivalent;
(2.1) f is lower semicontinuous on X.
(2.2) For all $\alpha \in \mathbb{R}$, the set $\{x|\phi(x) \leq \alpha\}$ is closed.
(2.3) for each $\varepsilon > 0$, there is a neighborhood U of x_0 such that

$$f(x) > f(x_0) - \varepsilon \quad \text{for all} \quad x \in U.$$

The following ordering principle is well-known and given in Goebel and Kirk[6].

Theorem II.2. *Let* (X, d) *be a complete metric space and* $\phi : X \rightarrow \mathbb{R}$ *a lower semi-continuous function which is bounded from below. Define a relation* $\leq$ *on* X*by*

$$x \leq y \Leftrightarrow d(x, y) \leq \phi(x) - \phi(y)$$

for all $x, y \in X$*. Then*
(1) $\leq$ *is an order relation on* X*,*
(2) ϕ *is nondecreasing, and*
(3) for each $x \in X$*, there is a maximal element* $v \in X$ *such that* $x \leq v$*.*

Using Theorem II.2, we can prove the following;

Theorem II.3 (Carsti-Kirk theorem). *Let* (X, d) *be a complete metric space and* $\phi : X \rightarrow \mathbb{R}$ *be a lower semi continuous function which is bounded from below. Suppose that* $f : M \rightarrow M$ *is an arbitrary mapping which satisfies*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)), x \in X.$$

Then f *has a fixed point.*

Note that any maximal element obtained from Theorem II.2 is indeed a fixed point of f in Theorem II.3.

Remark II.1. In theorem II.3, if we define $\phi(x) = \dfrac{1}{1-k} d(x, f(x))$ for all $x \in X$, then

$$d(x, f(x)) - kd(x, f(x)) \leq d(x, f(x)) - d(f(x), f^2(x))$$

imples that

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)), \quad x \in X.$$

Therefore Caristi theorem generalizes Banach contraction principle.

Park [11] and Kang [9] showed that Caristi-Kirk Theorem is equivalent to some maximal principles as follows;

Theorem II.4. *Let (X, d) be a complete metric space, and $\phi : X \to \mathbb{R}$ a lower semi-continuous function which is bounded from below. Then the following equivalent statements hold :*

(2.1) *X has a maximal element with respect to the order relation $\leq$ defined by*

$$x \leq y \Longleftrightarrow d(x, y) \leq \phi(x) - \phi(y), \quad x, y \in X.$$

(2.2) *If $f : X \to X$ is a selfmap saisfying*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)), \quad x \in X,$$

then f has a fixed point.

(2.3) *If* $F : X \to 2^X$ *is a multi-valued function satisfying*

$$\forall x \in X, \exists y \in F(x) \quad d(x, y) \leq \phi(x) - \phi(y)$$

has a fixed point.

Let (X, d) be a metric space and $x \in X$. We denote the open ball of radius r centered at x by $B(x, r)$. If A is a subset of X, we denote

$$B(A, r) = \bigcup_{x \in A} B(x, r).$$

We denote the set of all closed subsets of X by $C(X)$. Also, the set of all closed and bounded subsets of X is denoted by $CB(X)$. For two nonempty closed subsets $C, K \in C(X)$, we define the *Hausdorff metric* H by

$$H(C, K) = \inf\{\varepsilon > 0 : C \subseteq B(K, \varepsilon), K \subseteq B(C, \varepsilon)\},$$

whose value is in $[0, \infty]$.

Remark II.2. It is well-known that the above H is indeed a metric on the set $CB(X)$ of all closed and bounded subsets of X. Moreover, it is well-known that

$$H(C, K) = max\{\sup_{x \in C} \inf_{y \in K} d(x, y),\ \sup_{y \in K} \inf_{x \in C} d(x, y)\}.$$

The following is a generalization of Banach contraction principle to multivalued functions given by Nadler [10].

Theorem II.5 (Nadler). *Let* (X,d) *be a complete metric space and* $F : X \to 2^X$ *be a multivalued map. Suppose that there exists some* $k, 0 \leq k < 1$ *such that*

$$H(F(x),F(y)) \leq kd(x,y), \quad x,y \in X.$$

Then F *has a fixed point.*

Now we introduce the concept of fuzzy sets, fuzzy mappings and fuzzy fixed points.

Definition II.4. Let X be a nonempty set. A function $A : X \to [0,1]$ is called a *fuzzy set* in X and the set of all fuzzy sets in X is denoted by $F(X)$. If $A,B \in F(X)$, we define

$$A \subseteq B \;\Leftrightarrow\; A(x) \leq B(x),\; x \in X.$$

For any $x \in X$, the characteristic function $\chi_{\{x\}}$ is a fuzzy set and is denoted by $\{x\}$.

Let A$\in F(X)$ and $\alpha \in [0,1]$. The α level set of A, denote $[A]_\alpha$, is

$$[A]_\alpha = \{x \in X : A(x) \geq \alpha\},$$

if $\alpha \in (0,1]$ and

$$[A]_0 = \overline{\{x \in X : A(x) > 0\}}.$$

Let (X, d) be a metric space. We denote

$$FC(X) = \{A \in F(X) | [A]_1 \text{is nonempty and closed}\},$$

and

$$FW(X) = \{A \in F(X) | [A]_\alpha \text{is nonempty, closed and bounded for all } \alpha \in [0, 1]\}$$

Obviousely, we note that $FW(X) \subseteq FC(X)$.

For $A, B \in FC(X)$, we define

$$D_1(A, B) = H([A]_1, [B]_1),$$

and for $A, B \in FW(X)$, we define

$$D_\alpha(A, B) = H([A]_\alpha, [B]_\alpha),$$

for $\alpha \in [0, 1]$ and

$$D(A, B) = \sup_\alpha D_\alpha(A, B).$$

Definition II.5. A *fuzzy mapping* on X is a function from X into $F(X)$. For a fuzzy mapping T on X, an element x of X is called a *fixed point* of T if $\{x\} \subseteq T(x)$ i.e. $T(x)(x) = 1$.

Remark II.3. We may regard each fuzzy mapping $T : X \to F(X)$ as a two variable function of $X \times X$ into $[0, 1]$. So we will use the notation $T(x, y)$ instead of $T(x)(y)$ for $x, y \in X$.

III Main Results

In this section, we will prove some fixed point theorems for fuzzy mappings. To begin with, we have the following lemma due to Frigon and O'Regan [5];

Lemma III.1. *Let* (X, d) *be a complete metric space. Then* $(FW(X), D)$ *is a complete metric space.*

The following lemma is very important to prove our main results;

Lemma III.2. *Let* X *be a metric space and* $F : X \to C(X)$ *a continuous function where* $C(X)$ *is endowed with the Hausdorff metric.*
Then the function $\phi : X \to \mathbb{R}$ *defined by*

$$\phi(x) = d(x, F(x)) = \inf_{y \in F(x)} d(x, y)$$

is lower semi-continuous.

Proof. Let $\{x_n\}$ be a sequence in X with a limit $x \in X$. It suffices to show that $\liminf\limits_{n\to\infty} \phi(x_n) \geq \phi(x_n)$.

Let $a = \liminf\limits_{n\to\infty} \phi(x_n)$ and $\varepsilon > 0$ be given. Then for all $n \geq 1$, there is an index $m_n \geq n$ such that $\phi(x_{m_n}) < a + \varepsilon$, that is, $d(x_{m_n}, F(x_{m_n})) < a + \varepsilon$. So we can choose $y_{m_n} \in F(x_{m_n})$ such that $d(x_{m_n}, y_{m_n}) < a + \varepsilon$.

On the other hand, since $x_n \to x$ and F is continuous, there is an $N > 0$ such that $n \geq N$ implies $d(x_n, x) < \varepsilon$ and $H(F(x_n), F(x)) < \varepsilon$.

Let $n \geq N$. Since $m_n \geq n \geq N$, the last inequality shows that

$$\inf_{w \in F(x)} d(y_{m_n}, w) \leq \sup_{z \in F(x_{m_n})} \inf_{w \in F(x)} d(z, w) < \varepsilon.$$

Hence there is a $y \in F(x)$ such that $d(y_{m_n}, y) < \varepsilon$. Therefore

$$\begin{aligned} d(x, F(x)) &= \inf_{w \in F(x)} d(x, w) \\ &\leq d(x, y) \\ &\leq d(x, x_{m_n}) + d(x_{m_n}, y_{m_n}) + d(y_{m_n}, y) \\ &\leq a + 3\varepsilon. \end{aligned}$$

Since ε was arbitrary, $\phi(x) \leq a = \liminf_{n \to \infty} \phi(x_n)$.

Hence ϕ is lower semicontinuous.

Using Lemma III.2, we can prove the following fixed point theorem of Caristi type.

Theorem III.3. *Let (X, d) be a complete metric space, $T : X \to FC(X)$ be a fuzzy set. Suppose that there exists a lower semicontinuous function $\phi : X \to \mathbb{R} \cup \{\infty\}$ which is bounded from below. Assume that for any $x \in X$, there exists a $y \in X$ such that*

$$T(x, y) = 1, \quad \text{and} \quad d(x, y) \leq \phi(x) - \phi(y).$$

Then T has a fuzzy fixed point.

Proof. We define a multivalued function $F : X \to 2^X$.

$$F(x) = [T(x)]_1 = \{y \in X | T(x, y) = 1\}.$$

Then the above condition implies that for any $x \in X$, there exists $y \in F(x), \quad d(x, y) \leq \phi(x) - \phi(y)$.

Hence F satisfies the condition (2.3) in Theorem II.4. By Theorem II.4, F has a fixed point. That is, there exists an $x \in X$ such that

$$x \in F(x) = [T(x)]_1.$$

This shows that x is a fixed point of T.

Now we can prove Frigon and O'Regan's main theorem[5] by using Theorem III.3;

Theorem III.4. *Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $T : \overline{B(x_0, r)} \to FC(X) \quad (r > 0)$ *be a fuzzy mapping. Suppose there exists a constant* $k \in (0, 1)$ *with*

$$D_1(T(x), T(y)) \leq kd(x, y) \text{ for all } x, y \in \overline{B(x_0, r)}.$$

Then T *has a fuzzy fixed point. That is, there exists* $x \in \overline{B(x_0, r)}$ *with* $\{x\} \subseteq T(x),$ *i.e.,* $T(x, x) = 1$.

Proof. Let $x \in X$ and $p = \dfrac{1 + k}{2} \in (0, 1)$. We can choose a $y \in [T(x)]_1$ so that $d(x, y) \leq \dfrac{1 + p}{2p} d(x, [T(x)]_1)$. For this y, since

$$\inf_{z \in [T_y]_1} d(y, z) \leq D_1(T(x), T(y)) \leq kd(x, y) < pd(x, y),$$

there is a $z \in [T(y)]_1$ such that $d(y,z) \leq pd(x,y)$. Then we have

$$\begin{aligned} \frac{1+p}{p(1-p)}d(y,z) &\leq \frac{1+p}{1-p}d(x,y) \\ &= \left(\frac{1+p}{1-p}+1\right)d(x,y)-d(x,y). \end{aligned}$$

Hence

$$\begin{aligned} d(x,y) &\leq \frac{2}{1-p}d(x,y)-\frac{1+p}{p(1-p)}d(y,z) \\ &\leq \frac{1+p}{p(1-p)}d(x,[T(x)]_1)-\frac{1+p}{p(1-p)}d(y,[T(y)]_1). \end{aligned}$$

Hence the function $\phi(x) = \dfrac{1+p}{p(1-p)}d(x,[T(x)]_1)$ is lower semi-continuous by the above lemma for any $x \in X$, there exists a $y \in X$ such that

$$d(x,y) \leq \phi(x) - \phi(y).$$

By Theorem III.3, T has a fixed point.

From the above proof of Theorem III.4, we can state another fixed point theorem for fuzzy mapping as follows;

Theorem III.5. *Let (X,d) be a complete metric space and $T : X \to FC(X)$ be a fuzzy mapping. Suppose that there exists a $q, 0 < q < 1$ such that for any $x \in X$, there exists y such that*

$$T(x,y) = 1 \quad \text{and} \quad d(x,y) \leq qd(x,[T(x)]_1) - qd(y,[T(y)]_1).$$

Then T has a fixed point.

Proof. Define $\phi : X \to \mathbb{R}$ by

$$\phi(x) = qd(x, [T(x)]_1), \ x \in X.$$

Then ϕ is lower semicontinuous by lemma III.2. Hence the result following from Theorem III.3.

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ABSTRACT

Fixed Points of Fuzzy Contractive Maps

In 1969, Nadler generalized the Banach contraction principle for multi-valued maps.

Also in 1976, Caristi proved a fixed point theorem in complete metric spaces without assumming the continuity of given function. Caristi's Theorem generalizeds the Banach contraction principle.

Since Zadeh introduced the fuzzy set theory, a lot of structures on fuzzy sets are obtained and many authors have developed the fuzzy sets and their applications.

In 2002, Frigon and O'Regan proved a Banach type fixed point theorem for fuzzy mappings.

In this paper we prove some fixed point theorems for fuzzy contract on mappings. We show that fuzzy fixed point results can be deduced from the fixed point theory of multivalued mappings with closed values.

We show that Nadler's fixed point theorem for multi-valued contraction mappings can be deduced from Caristi's theorem.

Also We prove Frigon and O'Regan's Theorem using Caristi's theorem.

Moreover, we will show that some fixed point theorems for fuzzy mappings can be proved from the fixed point theorems for general multivalued mappings on complete metric spaces.