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姜秉介 教授指導  
碩士學位 請求論文

Fixed Points of Functions on  
Partially Ordered Metric Spaces

2008

誠信女子大學校 教育大學院

教育學科 數學教育專攻

姜 靜 銀

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# 認 准 書

姜 靜 銀의 碩士學位 論文을 認准함

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## 논문개요

순서집합(ordered set)에서 자기함수(self-function)의 부동점정리(Fixed point theorem)는 Zermelo의 정리로부터 시작되었다. Zermelo의 정리는 Zorn의 보조정리와 논리적으로 동치임은 잘 알려져 있는데, 이후로 많은 학자들이 순서집합에서 증가함수의 부동점의 존재에 흥미를 가지게 되었고, 순서집합에서의 극대원리와 부동점정리를 연계시키는 이론들이 나오게 되었다.

거리 공간에서는 Banach의 축소원리(Contraction Principle)는 1922년에 발표되었는데, 미분방정식 등 해석학분야에 대한 풍부한 응용력이 있어서 여러 가지 방향으로 일반화되었다. 1976년에 Caristi [2]는 주어진 함수에 연속성의 가정없이 완비 거리 공간에서 함수의 부동점정리를 증명하였다. Caristi는 그의 정리 증명에서 매우 복잡한 초한귀납법(transfinite induction)을 사용하였다. 2005년에 Nieto와 Lopez [10]는 제한된 영역에서 Banach형의 조건을 만족하는 연속 증가함수의 부동점정리를 증명하였는데, 그들의 증명은 함수의 반복열(iteration)의 수렴성을 이용하였다.

이 논문에서 우리는 주어진 함수의 연속을 가정하지 않고, Caristi형의 조건을 만족하는 증가함수에 대하여 Nieto와 Lopez의 부동점정리를 확장하였다. 또, Nieto와 Lopez의 조건 하에 몇 가지 다른 부동점정리를 증명하였다.

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# I Introduction

Let  $X$  be a nonempty set. If  $\leq$  is a relation on  $X$  which is reflexive, transitive and antisymmetric, then  $\leq$  is called an order relation on  $X$  and  $(X, \leq)$  is called a partially ordered set.

Fixed point theorems for functions on ordered sets are originated from the following theorem of Zermelo.

**Theorem I.1 (Zermelo).** *Let  $X$  be a partially ordered set and assume that every chain in  $X$  has a supremum. then every selfmap  $f : X \rightarrow X$  satisfying*

$$x \leq f(x), \quad x \in X$$

*has a fixed point.*

Since then, many authors gave interests to fixed points for increasing operators on ordered sets. It is well-known that some maximal principles can be reformulated to fixed point theorems in ordered sets. Indeed, Zermelo's Theorem is logically equivalent to Zorn's Lemma. See Park [12].

In 1955, Tarski [15] proved that every increasing function on a complete lattice has a fixed point. Also, Smithson [14] obtained a fixed point theorem for multivalued maps on partially ordered sets.

Let  $(X, d)$  be a metric space and  $f : X \rightarrow X$  be a function. Suppose that there exists a  $k \in \mathbb{R}$  such that

$$d(f(x), f(y)) \leq kd(x, y), \quad x, y \in X.$$

Then  $f$  is called a *contraction mapping*.

In metric spaces, the fixed point theorem generally known as the Banach's Contraction Principle appeared in 1922. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. The following is the Banach's Contraction Principle.

**Theorem I.2 (Banach's Contraction Principle).** *Let  $(X, d)$  be a complete metric space. Then every contraction mapping  $f : X \rightarrow X$  has a unique fixed point.*

Note that every contraction map is continuous. In 1976, Caristi [2] proved a fixed point theorem for functions on complete metric spaces without assuming the continuity of given function. Caristi used very complicated transfinite induction to prove his theorem. But now it is well-known that Caristi's Theorem can be deduced from a simple maximal principle on ordered sets.

**Theorem I.3 (Caristi).** *Let  $(X, d)$  be a complete metric space and  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous function which is bounded from below. Suppose that  $f : X \rightarrow X$  is an arbitrary mapping satisfying*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)), \quad x \in X.$$

*Then  $f$  has a fixed point.*

A lot of papers have been published to generalize or extend Banach's Contraction Principle and Caristi's fixed point Theorem. In 2005, Nieto and Lopez [10] prove a fixed point theorem for continuous increasing functions satisfying some Banach type conditions on some limited domain. He used the convergence of the iteration of given function to verify the existence of fixed point.

In this paper, we will extend Nieto and Lopez's a fixed point theorem for increasing functions satisfying some Caristi type conditions without assuming the continuity. We will use a maximality principle to prove our theorem.

In Section II, we will introduce some basic definitions and basic facts about ordered sets and ordered metric spaces.

In section III, we will prove our main theorems. Caristi type fixed point theorem under the condition of Nieto and Lopez [10]. Some other fixed point theorems will be given.

## II Preliminaries

In this section, we will show some definitions and preliminary results on ordered sets and ordered metric space.

In the following, let  $\mathbb{N}$  and  $\mathbb{R}$  denote the set of all natural numbers and the set of all real numbers, respectively.

Let  $X$  be a nonempty set. Then every function  $f : X \rightarrow X$  is called a *selfmap* of  $X$ . Let  $f : X \rightarrow X$  be a selfmap. If there is a point  $x \in X$  satisfying  $f(x) = x$ , then  $x$  is called a *fixed point* of  $f$ .

Let  $f : X \rightarrow X$  be a selfmap. Then we denote  $f^0(x) = x$  and  $f^2(x) = f(f(x))$  and  $f^n(x) = f(f^{n-1}(x))$  for all  $n \in \mathbb{N}$ . Each  $f^n$  is called the  *$n$ th iteration* of  $f$ .

**Definition II.1.** Let  $X$  be a nonempty set and  $2^X$  the power set of  $X$ . Then a function  $F : X \rightarrow 2^X$  is called a *multivalued function* on  $X$ .

Let  $F : X \rightarrow 2^X$  be a multivalued function on  $X$ , then we say that  $x_0 \in X$  is a *fixed point* of  $F$  if  $x_0 \in F(x_0)$ .

Let  $(X, \leq)$  be a partially ordered set. A function  $f : X \rightarrow X$  is said to be *monotone nondecreasing* if

$$x \leq y \Rightarrow f(x) \leq f(y), \quad x, y \in X.$$

For all  $x \in X$ , we denote

$$S(x) = \{y \in X \mid x \leq y\} \quad \text{and} \quad L(x) = \{y \in X \mid y \leq x\}.$$

For metric space  $(X, d)$  with an order relation  $\leq$ , we have the following;

**Definition II.2.** Let  $(X, d)$  be a metric space and  $\leq$  be an order relation on  $X$ . Then  $(X, d, \leq)$  is called an *ordered metric space* if  $\{x_n\}, \{y_n\}$  are convergent sequence in  $X$  and  $x_n \leq y_n$  for all  $n \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$ .

**Remark.** Let  $(X, d, \leq)$  be an ordered metric space and  $\{x_n\}$  be a convergent sequence in  $X$  such that  $x_n \leq y$ , then  $\lim_{n \rightarrow \infty} x_n \leq y$ .

The following basic property holds in ordered metric space.

**Proposition II.1.** *If  $(X, d, \leq)$  is an ordered metric space, then  $S(x)$  and  $L(x)$  are closed for all  $x \in X$ .*

**Proof.** Let  $\{y_n\}$  be a convergent sequence in  $S(x)$  and  $\lim_{n \rightarrow \infty} y_n = y_0$  then  $x \leq y_n$  for all  $n \in \mathbb{N}$  implies that  $x \leq \lim_{n \rightarrow \infty} y_n = y_0$ . Therefore,  $y_0 \in S(x)$ . Hence  $S(x)$  is closed. Similarly,  $L(x)$  is closed.

**Definition II.3.** Let  $(X, d)$  be a metric space. A function  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  is said to be *lower semicontinuous* if  $\{x \in X \mid \psi(x) \leq \alpha\}$  is closed for all  $\alpha \in \mathbb{R}$ .

Of course, every continuous map is lower semicontinuous. The following is a basic property of lower semicontinuous functions;

**Proposition II.2.** Let  $(X, d)$  be a metric space and  $x_0 \in X$ . If  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  is lower semicontinuous, then  $\psi(x_0) \leq \inf_{x \rightarrow x_0} \psi(x)$ .

**Proof.** Assume that  $\psi(x_0) > \liminf_{x \rightarrow x_0} \psi(x) = \alpha$ . Choose  $\varepsilon > 0$  so that  $\alpha < \alpha + \varepsilon < \psi(x_0)$ . Then for all  $n > 0$ , there exists an  $x_n \in X$  such that

$$x_n \in S = \{x \mid \psi(x) \leq \alpha + \varepsilon\} \text{ and } d(x_n, x_0) < \frac{1}{n} \text{ and } \psi(x_n) \leq \alpha + \varepsilon.$$

Since  $S$  is closed,  $\lim_{n \rightarrow \infty} x_n = x_0 \in S$ .

This shows that  $\psi(x_0) \leq \alpha + \varepsilon$ , which contradicts the fact that  $\alpha + \varepsilon \leq \psi(x_0)$ . Therefore  $\psi(x_0) \leq \liminf_{x \rightarrow x_0} \psi(x)$ .

If  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  is lower semicontinuous, then we can define a relation  $\leq$  on  $X$  by

$$x \leq y \Leftrightarrow d(x, y) \leq \psi(x) - \psi(y), \quad x, y \in X.$$

Then  $\leq$  is an order relation on  $X$  and  $(X, \leq)$  is a partially ordered set. Moreover, we have;

**Proposition II.3.** *Let  $(X, d)$  be a metric space and  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous function. If  $\leq$  is defined as above, then  $S(x)$  is closed for each  $x \in X$ .*

The following is a basic maximal principle on complete metric spaces.

**Theorem II.4.** *Let  $(X, d)$  be a complete metric space and  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous function which is bounded from below. Let  $\leq$  be the relation on  $X$  defined by*

$$x \leq y \Leftrightarrow d(x, y) \leq \psi(x) - \psi(y), \quad x, y \in X.$$

*Then  $\leq$  is an order relation on  $X$  and  $X$  has a maximal element.*

The proof of above theorem is given in Dugundji and Granas [3].

In 1976, Caristi [2] proved a fixed point theorem for functions on complete metric spaces without assuming continuity.

**Theorem II.5 (Caristi).** *Let  $(X, d)$  be a complete metric space and  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous function which is bounded from below. Suppose that  $f : X \rightarrow X$  is an arbitrary mapping satisfying*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)), \quad x \in X.$$

*Then  $f$  has a fixed point.*

Note that Caristi's original proof was based on transfinite induction. But if we use Theorem II. 4, the proof of Caristi's theorem is very simple. Suppose that  $x_0 \in X$  is maximal. Then  $d(x_0, f(x_0)) \leq \psi(x_0) - \psi(f(x_0))$  implies that  $x_0 \leq f(x_0)$ . By the maximality of  $x_0$ ,  $x_0 = f(x_0)$ .

**Remark.** Assume that  $d(x, f(x)) \leq \psi(x) - \psi(f(x))$  for all  $x \in X$ , then  $x \leq f(x)$  for all  $x \in X$  and  $f$  is a monotone function.

### III Main Results

In this section, we will prove some fixed point theorems for functions satisfying Caristi type conditions.

In 2005, Nieto and Lopez [10] proved Banach type fixed point theorem to extend the theoretical results of fixed points in partially ordered sets and apply them to study a problem of ordinary differential equations.

The following Caristi type fixed point theorem generalizes the theorem of Nieto and Lopez.

**Theorem III.1.** *Let  $(X, d, \leq)$  be a complete ordered metric space. Let  $f : X \rightarrow X$  be a nondecreasing function and that there is a lower semicontinuous function  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  which is bounded below and*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x))$$

*for all  $x \in X$  satisfying  $x \leq f(x)$ . Assume that  $A = \{x \mid x \leq f(x)\}$  is nonempty and closed, then  $f$  has a fixed point.*

**Proof.** If  $f(x_0) = x_0$ , the proof is finished. If  $y \in A$ , then  $y \leq f(y)$ . since  $f$  is nondecreasing,  $f(y) \leq f(f(y))$  implies that  $f(y) \in A$ . So  $f$  maps  $A$  into  $A$ .

Define  $\preceq$  on  $X$  by

$$x \preceq y \Leftrightarrow x \leq y \text{ and } d(x, y) \leq \psi(x) - \psi(y).$$

Then  $\preceq$  is an order relation on  $X$ . Note that

$$S(x) = \{y \mid x \preceq y\} = \{y \mid x \leq y \text{ and } d(x, y) \leq \psi(x) - \psi(y)\}$$

is closed in  $X$ . Since  $x \leq f(x)$  for all  $x \in A$ ,

$$d(x, f(x)) \leq \psi(x) - \psi(f(x))$$

for all  $x \in A$ . Hence  $x \preceq f(x)$  for all  $x \in A$ . Choose  $x_0 \in A$  and let

$$a = \inf_{x \in S(x_0)} \psi(x).$$

For  $n = 1$ , there exists an  $x_1 \in S(x_0)$ ,  $a \leq \psi(x_1) < a + 1$ . Assume that  $x_1 \leq x_2 \leq \cdots \leq x_n$  were chosen so that

$$\inf_{x \in S(x_{n-1})} \psi(x) \leq \psi(x_n) < \inf_{x \in S(x_{n-1})} \psi(x) + \frac{1}{n}. \quad (1.1)$$

Choose an  $x_{n+1} \in S(x_n)$  such that

$$\inf_{x \in S(x_n)} \psi(x) \leq \psi(x_{n+1}) < \inf_{x \in S(x_n)} \psi(x) + \frac{1}{n+1}. \quad (1.2)$$

Inductively, we can construct a sequence  $x_n$  satisfying (1.2). Then  $\{S(x_n)\}$  is a decreasing sequence of closed sets in  $A$ .

Assume that  $y \in S(x_n)$ , then  $y \in S(x_n) \subseteq S(x_{n-1})$  and

$$\psi(y) \geq \inf_{x \in S(x_{n-1})} \psi(x) \geq \psi(x_n) - \frac{1}{n}.$$

$$\therefore \psi(x_n) - \psi(y) \leq \frac{1}{n}$$

Hence if  $y \in S(x_n)$ , then

$$d(x_n, y) \leq \psi(x_n) - \psi(y) \leq \frac{1}{n}.$$

Therefore  $\text{diam}S(x_n) \leq \frac{2}{n}$  for all  $n \in \mathbb{N}$ . So  $\{S(x_n)\}$  is a decreasing sequence of closed sets in  $A$  such that  $\text{diam}S(x_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Since  $X$  is complete,  $\bigcap_{n=1}^{\infty} S(x_n) = \{\xi\}$  for some  $\xi \in A$ .  $\xi \in S(x_n)$  implies that  $x_n \leq \xi$  for all  $n \in \mathbb{N}$ .

If  $\xi \leq \eta$  in  $X$ , then  $\eta \geq \xi \geq x_n$  implies that  $\eta \in S(x_n)$  for all  $n \in \mathbb{N}$ . So

$$\eta \in \bigcap_{n=1}^{\infty} S(x_n) = \{\xi\} \quad \text{and} \quad \text{hence} \quad \xi = \eta.$$

Hence  $\xi$  is a maximal element in  $(X, \preceq)$ . But then  $\xi \leq f(\xi)$  implies that  $d(\xi, f(\xi)) \leq \psi(\xi) - \psi(f(\xi))$ . Hence  $\xi \preceq f(\xi)$  and since  $\xi$  is maximal with respect to  $\preceq$ ,  $\xi = f(\xi)$ .

**Remark.** Assume that  $f$  is continuous and  $\leq$  is closed. Since

$$d(x_n, \xi) \leq \text{diam}S(x_n) \leq \frac{2}{n}$$

for all  $n \in \mathbb{N}$ ,  $\{x_n\}$  converges to  $\xi$ . If  $f$  is continuous, then  $x_n \leq f(x_n)$  implies that

$$\xi = \lim_{x \rightarrow \infty} x_n \leq \lim_{x \rightarrow \infty} f(x_n) = f(\xi).$$

That is,  $\xi \leq f(\xi)$ .

We can prove Nieto and Lopez's Theorem [10] by using Theorem III.1.

**Corollary 1 (Nieto and Lopez).** *Let  $(X, \leq)$  be a partially ordered set and suppose that there exists a metric  $d$  in  $X$  such that  $(X, d)$  is a complete metric space. Let  $f : X \rightarrow X$  be a continuous and nondecreasing mapping such that there exists  $k \in [0, 1)$  with*

$$d(f(x), f(y)) \leq kd(x, y)$$

*for all  $x, y$  satisfying  $x \geq y$ . If there exists  $x_0 \in X$  with  $x_0 \leq f(x_0)$ , then  $f$  has a fixed point.*

**Proof.** Let  $\psi(x) = \frac{1}{1-k}d(x, f(x))$  is continuous for  $x \in X$ . Then

$$d(x, f(x)) - kd(x, f(x)) \leq d(x, f(x)) - d(f(x), f^2(x))$$

thus

$$d(x, f(x)) \leq \frac{1}{1-k}d(x, f(x)) - \frac{1}{1-k}d(f(x), f^2(x)).$$

Hence

$$d(x, f(x)) \leq \psi(x) - \psi(f(x))$$

for all  $x \in X$  satisfying  $x \leq f(x)$ .

Therefore, Corollary 1 follows from Theorem III.1.

**Remark.** Let  $x_n = f^n(x_0)$  for  $x_0 \in X$  and  $n \in \mathbb{N}$ . Then

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \leq \psi(x_n) - \psi(x_{n+p})$$

thus

$$0 \leq d(x_n, x_{n+1}) \leq \psi(x_n) - \psi(x_{n+1}).$$

Hence  $\psi(x_n)$  is decreasing. Since

$$\sum_{i=n}^{n+p} d(x_i, x_{i+1}) \leq \psi(x_n) - \psi(x_{n+p}).$$

In particular,  $\sum_{i=0}^{\infty} d(x_i, x_{i+1})$  is converges. Hence  $\{x_n\}$  is a Cauchy sequence. So  $\{f^n(x_0)\}$  converges to a fixed point  $x$  of  $f$ .

We have proved a fixed point theorem of Caristi type on complete ordered metric spaces. The following is a fixed point theorem for continuous nondecreasing functions on metric space.

**Theorem III.2.** *Let  $(X, d, \leq)$  be a complete ordered metric space. Let  $f : X \rightarrow X$  be a continuous nondecreasing function and that there is a lower semicontinuous function  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  which is bounded below and*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x))$$

*for all  $x \in X$  satisfying  $x \leq f(x)$ . Assume that there exists an  $x_0 \in X$  such that  $x_0 \leq f(x_0)$ . Then  $f$  has a fixed point.*

**Proof.** Since  $x_0 \leq f(x_0)$ ,

$$d(x_0, f(x_0)) \leq \psi(x_0) - \psi(f(x_0))$$

by assumption. Since  $f$  is increasing,  $x_1 = f(x_0) \leq f(x_1)$ . Let  $x_1 = f(x_0)$  and assume that  $x_1, x_2, \dots, x_n$  were chosen so that

$$x_k \leq x_{k+1}, \quad x_k \leq x_{k+1} = f(x_k), \quad k = 0, 1, \dots, n-1.$$

Then  $d(x_{n-1}, x_n) \leq \psi(x_{n-1}) - \psi(x_n)$  and  $x_n \leq f(x_{n-1}) \leq f(x_n)$ . Let  $x_{n+1} = f(x_n)$ . By induction, we can construct a sequence  $\{x_n\}$  such that  $x_n \leq x_{n+1} = f(x_n)$  and

$$d(x_n, x_{n+1}) \leq \psi(x_n) - \psi(x_{n+1}), \quad n = 1, 2, \dots.$$

$\{\psi(x_n)\}$  is a real sequence bounded below decreasing. Hence  $\alpha = \inf_{n \in \mathbb{N}} \psi(x_n)$  exists. For any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $\psi(x_n) < \alpha + \varepsilon$ . Assume that  $m \geq n \geq N$  then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \psi(x_n) - \psi(x_{n+1}) + \psi(x_{n+1}) - \psi(x_{n+2}) + \dots + \psi(x_{m-1}) - \psi(x_m) \\ &\leq \psi(x_n) - \psi(x_m) \leq \alpha + \varepsilon - \alpha = \varepsilon. \end{aligned}$$

Hence  $\{x_n\}$  is a Cauchy sequence. So  $\{x_n\}$  converges to some  $\xi \in X$ .

Since  $f$  is continuous,

$$f(\xi) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = \xi.$$

Therefore,  $\xi$  is a fixed point of  $f$ .

We have another fixed point theorem under the condition given in Nieto and Lopez [10];

**Theorem III.3.** *Let  $(X, \leq)$  be a partially ordered set and suppose that there exists a metric  $d$  in  $X$  such that  $(X, d)$  is a complete metric space and  $\leq$  is a closed order. Let  $f : X \rightarrow X$  be a continuous nondecreasing function such that there is a lower semicontinuous function  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  which is bounded from below and*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x))$$

*for all  $x \in X$  satisfying  $x \leq f(x)$ . If there exists a point  $x_0 \in X$  such that  $x_0 \leq f(x_0)$ , then  $f$  has a fixed point.*

**Proof.** As in the proof of Theorem III.2, we can show that  $\{S(x_n)\}$  is a decreasing sequence of closed sets in  $X$ . Hence there exists a  $\xi \in X$  such that  $\bigcap_{n=1}^{\infty} S(x_n) = \{\xi\}$ .

We will show that  $\xi \leq f(\xi)$ . Since  $\text{diam}S(x_n) \leq \frac{2}{n}$  for all  $n \in \mathbb{N}$ ,  $\{x_n\}$  converges to  $\xi$ .

Hence  $\xi = \lim_{x \rightarrow \infty} x_n \leq \lim_{x \rightarrow \infty} f(x_n) = f(\xi)$ . That is,  $\xi \leq f(\xi)$ . Since  $\xi$  is maximal element in  $(X, \leq)$ ,  $\xi = f(\xi)$ .

For multivalued function  $F : X \rightarrow 2^X$ , we have the following;

**Theorem III.4.** *Let  $(X, d, \leq)$  be a complete ordered metric space. Let  $F : X \rightarrow 2^X \setminus \{\emptyset\}$  be a multivalued function. Suppose that there exists a lower semicontinuous function  $\psi : X \rightarrow \mathbb{R} \cup \{\infty\}$  which is bounded below and*

$$\forall x \in X, \exists y \in F(x), x \leq y \text{ and } d(x, y) \leq \psi(x) - \psi(y).$$

Assume that  $A = \{x \in X \mid x \leq y \text{ for some } y \in F(x)\}$  is nonempty and closed, then  $F$  has a fixed point.

**Proof.** As in the proof of Theorem III.1, the relation  $\preceq$  defined by

$$x \preceq y \Leftrightarrow x \leq y \text{ and } d(x, y) \leq \psi(x) - \psi(y)$$

is an order relation and there exists a maximal element  $\xi$  in  $A$ . By assumption, there exists an  $\eta \in F(\xi)$  such that  $\xi \leq \eta$  and

$$d(\xi, \eta) \leq \psi(\xi) - \psi(\eta).$$

That is,  $\xi \preceq \eta$  and by the maximality of  $\xi$ ,  $\xi = \eta \in F(\xi)$ .

Hence  $\xi$  is a fixed point of  $F$ .

# References

- [1] F. H. Croom, *Principles of Topology*, Philadelphia: Saunders College Pub, (1989).
- [2] J. Carsti, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. soc. 215, 241-51, (1976).
- [3] J. Dugundji and A. Granas, *Fixed Point Theory*, Polish Scientific Publishers, Warszawa, (1982).
- [4] N. Dunford and J. Schwartz, *Linear operators Part I*, Interscience Pub (1958), 4-7.
- [5] K. Goebel and W.A Kirk, *Topics in metric fixed point theory*, Cambridge Univ. Press, (1990).
- [6] B. G. Kang, *Fixed points of increasing operators and maximal principles*, 성신연구논문집, 37 (2001), 219-228.
- [7] B. G. Kang, *Muti-valued fixed points theorems in ordered sets*, Sungshin Women's University Journal of Basic Science, Vol. 4. (1987)

- [8] S. L Lee, *Fixed points of increasing operators on ordered spaces* , 성신 여자대학교 교육대학원 수학교육전공, (2006)
- [9] M. J. Lee, *Fixed points of increasing multivalued maps* , 성신여자대학교 교육대학원 수학교육전공, (2006)
- [10] J.J Nieto, R. Rodriguez-Lopez, *Contractive Mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22 (3) (2005) 223-229.
- [11] S. Park, *Fixed points of multimaps on ordered spaces*, Math. Sci. Res. Hot-Line 2 (9) (1998), 9-14.
- [12] S. Park, *Equivalent formulations of Zorn's lemma and other maximum principles*, J. Korea Soc. Math. Ed. 25 (1987), 19-24.
- [13] Ran, A.C.M. and Reurings, M.C.B *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Am. Math. Soc. 132 (2004), 1435-1443.
- [14] R. E. Smithson, *Fixed points of order preserving multifunctions*, Proc. Amer. Math. Soc. 28(1)(1971), 304-310.
- [15] A. Tarski, *A lattice theoretic fixpoint theorems and its applications*, Pacific J. Math., 5 (1955), 285-310.

## Abstract

Fixed point theorems for functions on ordered sets are originated from the theorem of Zermelo. It is well-known that some maximal principles can be reformulated to fixed point theorems in ordered sets. Indeed, Zermelo's Theorem is logically equivalent to Zorn's Lemma.

In metric spaces, the fixed point theorem generally known as the Banach's Contraction Principle appeared in 1922. In 1976, Caristi [2] proved a fixed point theorem for functions on complete metric spaces without assuming the continuity of given function. Caristi used very complicated transfinite induction to prove his theorem. Since then, a lot of papers have been published to generalize or extend Banach's Contraction Principle and Caristi's fixed point Theorem. In 2005, Nieto and Lopez [10] prove a fixed point theorem for continuous increasing functions satisfying some Banach type conditions on some limited domain. He used the convergence of the iteration of given function to verify the existence of fixed point.

In this paper, We will extend Nieto and Lopez's a fixed point theorem for increasing functions satisfying some Caristi type conditions without assuming the continuity. We will use a maximality principle to prove our theorem. Also we will show that another fixed point theorem under the condition given in Nieto and Lopez.