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석사학위 청구논문

Artinition  $O$ -sequences of  
Socle Degree 5 and Type 3

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성신여자교육대학교 교육대학원  
교육학과 수학교육전공  
조 정 연

# Artinian $O$ -sequences of Socle Degree 5 and Type 3

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이 논문을 석사학위 논문으로 제출함.

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# 인 준 서

조 정 연의 석사학위 논문으로 인준함.

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## 논문개요

Fröberg and Laksov 의 정리에 근거하여 여차원이 3, 형태가 3, 길이가 6인 level이 될 가능성이 있는 73개의 Artinian O-수열에 대하여 level 여부를 연구하였다.

우선 참고문헌 D.S. Shin and Y.S. Shin, J.M. Ahn and Y.S. Shin, A.V. Geramita, T. Harima, J.C. Migliore and Y.S. Shin 을 이용하여 level이 될 가능성이 없는 39개를 제외하였다.

또한 참고문헌 A.V. Geramita, T. Harima, J.C. Migliore and Y.S. Shin을 이용하여 2개의 수열은 level이 됨을 보였다. 그리고 남은 32개의 level이 되는 수열은 CoCoA, S-plus를 이용하여 “link-sum”을 구성하였다.

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## Abstract

# ABSTRACT

## Artinian $\mathcal{O}$ -sequences of Socle Degree 5 and Type 3

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The purpose of this study was to examine the Codimension 3, Socle Degree 5 and Type 3 Artinian  $\mathcal{O}$ -sequences based on Theorem of Fröberg and Laksov.

First of all, we prove that 39 sequences cannot be level using [1], [4], [14], and we prove that 2 sequences can be level by Theorem in [4].

Then, we introduce how to construct 32 sequences using CoCoA, S-plus.

## 1. Introduction

Let  $k$  be an infinite field,  $R = k[x_1, \dots, x_n] = \bigoplus_{i \geq 0} R_i$ , and let  $I$  be a homogeneous ideal of  $R$ ,  $A = R/I$ . The *Hilbert function of  $A$* ,  $\mathbf{H}_A : \mathbb{N} \rightarrow \mathbb{N}$ , defined by,

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t$$

has been much studied and, in case  $I$  is the ideal of a subscheme of  $\mathbb{P}^{n-1}$ , contains a great deal of information about the geometry of this subscheme.

Let  $\mathbf{H}$  be a numerical function which can be the Hilbert function of a finite set of points in  $\mathbb{P}^n$ . For any particular  $\mathbf{H}$ , there are infinitely many different collections of points which share  $\mathbf{H}$  as their Hilbert function. In [5], they showed, that there is a one to one correspondence between all possible Hilbert functions of finite sets of points in  $\mathbb{P}^n$  and  $n$ -type vectors (see Theorem 2.6, [5], see also [6] or [7]).

In [8], they asked the following questions.

- (1) If  $\mathbf{H}$  is a possible Hilbert function for a set of reduced points in  $\mathbb{P}^n$ , what are all the possible graded Betti numbers for sets of points  $\mathbb{X} \subseteq \mathbb{P}^n$  which have Hilbert function  $\mathbf{H}$ ?
- (2) What can be the Hilbert function of a level Artinian algebra, (see Section 2 for the definition)?

By works of Bigatti [13], Hulett [11], and Pardue [12], we know that, given  $\mathbf{H}$ , there is a unique maximal set of graded Betti numbers which can have.

In Section 2, we introduce some definitions and preliminary observations about Gorenstein and level Artinian algebras (see also [8]). In [8], they used the fact that if a set of points in  $\mathbb{P}^2$  has level coordinate ring, then one can break the set up two subsets and used those subsets to construct a level algebra of codimension 3. Moreover, they used this method to get a complete list (in codimension 3) of all Hilbert functions of level algebras of type 2 and socle degree  $\leq 5$ . Recently, in [4], they obtained all possible Hilbert functions (in codimension 3) of level algebras of socle degree 6 and type 2.

In Section 3, we prove that some special O-sequences cannot be level (see Propositions 3.8 and 3.10)

In Section 4, we introduce how to construct all codimension 3 level Artinian algebras of socle degree 5 and type 3.

## 2. Gorenstein and Level O-sequences

**Definition-Proposition 2.1** (Definition-Proposition 2.21, [8]). Let  $R = k[x_0, \dots, x_n]$  and let  $A = R/I$  be a Cohen-Macaulay ring of dimension  $d$ . Let

$$0 \rightarrow \mathcal{F}_{n-(d-1)} \rightarrow \cdots \rightarrow \mathcal{F}_1 \rightarrow I \rightarrow 0$$

be a minimal free resolution of  $I$ . Then

- (a) If  $B = B_0 \oplus \cdots \oplus B_\ell$  ( $B_\ell \neq 0$ ) is an Artinian algebra, then  $B$  is **level** if and only if  $B_\ell = \text{Ann}(B_1)$ .
- (b)  $A$  is a **level algebra** if  $\mathcal{F}_{n-(d-1)} = R^m(-s)$ , for some  $s > 0$ .  
 $\text{rank } \mathcal{F}_{n-(d-1)}$   
 $=$  Cohen-Macaulay type of  $A$ .
- (c) i) If  $\mathbb{X}$  is a non-degenerate set of points in  $\mathbb{P}^n$ ,  $A = R/I_{\mathbb{X}}$  its coordinate ring, then we say that  $\ell$  is the **socle degree** of  $\mathbb{X}$  if  $\ell$  is the socle degree of the Artinian algebra  $B = A/\overline{L}A$ , where  $\overline{L}$  is any linear non-zero-divisor of  $A$ .
- ii)  $\mathbb{X}$  is called a **level set** of points if  $A = R/I_{\mathbb{X}}$  is a level algebra. In this case, the socle degree of  $\mathbb{X}$  is  $\ell = \sigma(\mathbb{X}) + n - 1$ .
- (d) If  $\overline{L}$  is a linear non-zero divisor in  $A = R/I$ , then  $A$  is level if and only if  $A/\overline{L}A \simeq A/(L, I_{\mathbb{X}})$  is level.
- (e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an Artinian algebra, see e.g., [9])  $b = \{b_i\}_{i \geq 0}$  with  $b_1 = n + 1$ , is called **level** if there is a level set of points in  $\mathbb{P}^n$  with Hilbert function  $b$ .

Here we have an interesting question.

**Question 2.2.** What are the possible Hilbert functions of level Artinian algebras of codimension 3 with  $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = 3$ ?

### 3. Non existence of Level O-sequences of Socle Degree 5 and Type 3

Before we construct all possible level Artinian O-sequences of socle degree 5 and type 3, we would like to introduce how to construct *all* Gorenstein O-sequences of socle degree 5 in the following Example 3.1 using Corollary 4.6. Notice that we used CoCoA [2], a system for doing **C**omputations in **C**ommutative **A**lgebra to calculate the Hilbert functions in Example 3.1.

**Example 3.1.** Consider any level O-sequence  $\mathbf{H} : 1 \ 3 \ \alpha \ \beta \ \gamma \ 1 \ 0$ . Since the sequence  $\mathbf{H}$  is of type 1, we have that  $\mathbf{H}$  is a Gorenstein sequence of socle degree 5, and hence we have that

$$\alpha = \beta = 3, 4, 5, \text{ or } 6 \text{ and } \gamma = 3.$$

Let  $\mathbb{X}$  be the set of all  $\bullet$ 's in  $\mathbb{Z}$  and  $\mathbb{Y}$  be the set of all  $*$ 's in  $\mathbb{Z}$ .

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & \bullet & \bullet \\ * & * & \bullet & \bullet \\ * & \bullet & * & \bullet \\ * & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 13 & 15 & 16 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 6 & 3 & 1 & 0 & \rightarrow . \end{aligned}$$

By the same idea, we can construct all the other cases when  $\alpha = \beta = 3, 4,$  or,  $5$ .

We recall a special case of a theorem of Fröberg and Laksov of [3] that if  $A$  is a level Artinian algebra of codimension 3 and type  $m$  with  $\sigma - 1 = t$ , then

$$\mathbf{H}(A, i) \leq \min \left\{ \binom{2+i}{2}, m \binom{t-i+2}{2} \right\}.$$

The following algorithm for CoCoA is due to Georgio Daldozzo in Italy.

```

Define HvectorAdmis2(K)
  L := [[1,3]];
  For I:=2 To K Do
    Prov := [];
    Foreach El In L[I-1] Do
      For J:=1 To BinExp(El[Len(El)],I-1,1,1) Do
Append(Prov,Concat(El,[J]));
    EndFor;
  EndForeach;
Append(L,Prov);

```

```

EndFor;
A := L[Len(L)];
B:= [Elements In A | Elements[Len(Elements)]=2];
Return [Elements In B | Elements[Len(Elements)-1]< 7];
EndDefine;

```

**Remark 3.2.** Consider the sequence  $\mathbf{H} : 1 \ 3 \ \alpha \ \beta \ \gamma \ 2 \ 0$ . Using the above algorithm for CoCoA, we obtained 79 possible Artinian O-sequences of type 2 (see Example 3.18, [8]). The only 24 cases are all possible level Artinian O-sequences among all 79 possible Artinian O-sequences.

Here we introduce how to construct the most interesting case

$$1 \ 3 \ 6 \ 10 \ 6 \ 2$$

among 24 possible level Artinian O-sequences, which is extremal.

$$\mathbb{Z} = \begin{cases} * & \bullet \\ * & \bullet \\ * & \bullet & \bullet & * \\ \bullet & * & * & * \\ \bullet & * & \bullet & * \\ \bullet & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 20 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 10 \ 10 \ 10 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 10 \ 6 \ 2 \ 0 \ \rightarrow . \end{aligned}$$

We now consider Question 2.2 for the case  $\sigma = 6$ . Using the above Georgio's algorithm for CoCoA again, we obtain all possible 73 Artinian O-sequences of type 3 as in Table 1.

1)	1, 3, 3, 3, 3, 3	2)	1, 3, 3, 4, 3, 3	3)	1, 3, 3, 4, 4, 3
4)	1, 3, 3, 4, 5, 3	5)	1, 3, 4, 3, 3, 3	6)	1, 3, 4, 4, 3, 3
7)	1, 3, 4, 4, 4, 3	8)	1, 3, 4, 4, 5, 3	9)	1, 3, 4, 5, 3, 3
10)	1, 3, 4, 5, 4, 3	11)	1, 3, 4, 5, 5, 3	12)	1, 3, 4, 5, 6, 3
13)	1, 3, 5, 3, 3, 3	14)	1, 3, 5, 4, 3, 3	15)	1, 3, 5, 4, 4, 3
16)	1, 3, 5, 4, 5, 3	17)	1, 3, 5, 5, 3, 3	18)	1, 3, 5, 5, 4, 3
19)	1, 3, 5, 5, 5, 3	20)	1, 3, 5, 5, 6, 3	21)	1, 3, 5, 6, 3, 3
22)	1, 3, 5, 6, 4, 3	23)	1, 3, 5, 6, 5, 3	24)	1, 3, 5, 6, 6, 3
25)	1, 3, 5, 6, 7, 3	26)	1, 3, 5, 7, 3, 3	27)	1, 3, 5, 7, 4, 3
28)	1, 3, 5, 7, 5, 3	29)	1, 3, 5, 7, 6, 3	30)	1, 3, 5, 7, 7, 3
31)	1, 3, 5, 7, 8, 3	32)	1, 3, 5, 7, 9, 3	33)	1, 3, 6, 3, 3, 3
34)	1, 3, 6, 4, 3, 3	35)	1, 3, 6, 4, 4, 3	36)	1, 3, 6, 4, 5, 3
37)	1, 3, 6, 5, 3, 3	38)	1, 3, 6, 5, 4, 3	39)	1, 3, 6, 5, 5, 3
40)	1, 3, 6, 5, 6, 3	41)	1, 3, 6, 6, 3, 3	42)	1, 3, 6, 6, 4, 3
43)	1, 3, 6, 6, 5, 3	44)	1, 3, 6, 6, 6, 3	45)	1, 3, 6, 6, 7, 3
46)	1, 3, 6, 7, 3, 3	47)	1, 3, 6, 7, 4, 3	48)	1, 3, 6, 7, 5, 3
49)	1, 3, 6, 7, 6, 3	50)	1, 3, 6, 7, 7, 3	51)	1, 3, 6, 7, 8, 3
52)	1, 3, 6, 7, 9, 3	53)	1, 3, 6, 8, 3, 3	54)	1, 3, 6, 8, 4, 3
55)	1, 3, 6, 8, 5, 3	56)	1, 3, 6, 8, 6, 3	57)	1, 3, 6, 8, 7, 3
58)	1, 3, 6, 8, 8, 3	59)	1, 3, 6, 8, 9, 3	60)	1, 3, 6, 9, 3, 3
61)	1, 3, 6, 9, 4, 3	62)	1, 3, 6, 9, 5, 3	63)	1, 3, 6, 9, 6, 3
64)	1, 3, 6, 9, 7, 3	65)	1, 3, 6, 9, 8, 3	66)	1, 3, 6, 9, 9, 3
67)	1, 3, 6, 10, 3, 3	68)	1, 3, 6, 10, 4, 3	69)	1, 3, 6, 10, 5, 3
70)	1, 3, 6, 10, 6, 3	71)	1, 3, 6, 10, 7, 3	72)	1, 3, 6, 10, 8, 3
73)	1, 3, 6, 10, 9, 3				

TABLE 1. All possible Artinian O-sequences of length 6 and type 3

Before we find all possible level Artinian O-sequences of socle degree 5 and type 3, we need the following theorems and corollary to eliminate some of non-level cases.

**Theorem 3.3** (Theorem 2.12, [4]). *Let  $h_{d-2}$ ,  $h_{d-1}$ ,  $h_d$  be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d \rangle} \quad \text{and} \quad h_{d-1} = h_{d-2}^{\langle d-1 \rangle}.$$

*Let  $I$  be any ideal in  $R = k[x_1, \dots, x_n]$  such that the Hilbert function of  $R/I$  satisfies*

$$\begin{aligned} \mathbf{H}(R/I, d-2) &= h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) &= h_{d-1} \\ \mathbf{H}(R/I, d) &= h_d. \end{aligned}$$

*Then, the ring  $R/I$  has socle of dimension  $\varepsilon$  in degree  $d-2$ .*

**Theorem 3.4** (Theorem 8, [14]). *If  $\mathbf{H} : h_0 \ h_1 \ \cdots \ h_t \ 0 \rightarrow$  is a level Artinian O-sequence where  $t \geq 2$ , then  $\mathbf{G} : h_0 \ h_1 \ \cdots \ h_{t-1} \ 0 \rightarrow$  is also a level Artinian O-sequence.*

Using Theorems 6 and 7, the following cases in Table 2 are not level.

2)	1, 3, 3, 4, 3, 3	3)	1, 3, 3, 4, 4, 3	4)	1, 3, 3, 4, 5, 3
6)	1, 3, 4, 4, 3, 3	8)	1, 3, 4, 4, 5, 3	9)	1, 3, 4, 5, 3, 3
14)	1, 3, 5, 4, 3, 3	16)	1, 3, 5, 4, 5, 3	17)	1, 3, 5, 5, 3, 3
20)	1, 3, 5, 5, 6, 3	21)	1, 3, 5, 6, 3, 3	26)	1, 3, 5, 7, 3, 3
34)	1, 3, 6, 4, 3, 3	36)	1, 3, 6, 4, 5, 3	37)	1, 3, 6, 5, 3, 3
40)	1, 3, 6, 5, 6, 3	41)	1, 3, 6, 6, 3, 3	46)	1, 3, 6, 7, 3, 3
53)	1, 3, 6, 8, 3, 3	60)	1, 3, 6, 9, 3, 3	67)	1, 3, 6, 10, 3, 3

TABLE 2. By Theorems 6 and 7, the cases are not level.

**Theorem 3.5** (Theorem 4.1, [1]). *Let  $R = k[x_1, x_2, x_3]$  and let  $\mathbf{H} = (h_0, h_1, \dots, h_s)$  be the  $h$ -vector of a graded Artinian algebra  $A = R/I$  with socle degree  $s$ . If*

$$h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \leq 2d + 3,$$

*then  $\mathbf{H}$  is not level*

5)	1, 3, 4, 3, 3, 3	13)	1, 3, 5, 3, 3, 3	15)	1, 3, 5, 4, 4, 3
33)	1, 3, 6, 3, 3, 3	35)	1, 3, 6, 4, 4, 3	39)	1, 3, 6, 5, 5, 3

TABLE 3. By Theorem 8, the cases are not level.

**Theorem 3.6** (Corollary 2.11, [4]). *Let  $h = (1, h_1, \dots, h_s)$  be a level sequence with  $h_s \geq 1$  and let  $A$  be any Artinian algebra with  $\mathbf{H}_A = h$ .*

*Then the vector  $(h_s, h_{s-1}, \dots, h_1, 1)$  is the sum of:*

- 1) *a Gorenstein sequence  $(1, b_1, \dots, b_{s-1}, 1)$  (of length  $s+1$ ) which is the  $h$ -vector of a quotient of  $A$  and*
- 2)  *$h_s - 1$   $O$ -sequences  $(1, d_{j1}, \dots, d_{jr_j})$ ,  $j = 1, \dots, h_s - 1$ , each of which is the  $h$ -vector of a quotient of  $A$ .*

**Remark 3.7** (Remark 2.7, [4]). There is one more additional useful fact that comes out of the proof. If we let  $\tilde{h} = (1, \tilde{h}_1, \dots, \tilde{h}_r)$ ,  $r \leq s$  be any one of the O-sequences in the decomposition promised by Theorem 3.6, then  $\tilde{h}_1 \leq h_1$ .

**Proposition 3.8.** *There is no level Artinian algebra with the Hilbert function 1 3 6 8 4 3.*

*Proof.* Assume 1 3 6 8 4 3 is level. Then the reverse of (1, 3, 6, 8, 4, 3) can be written as the sum of a Gorenstein sequence (1,  $b_1, \dots, b_4, 1$ ) and an two O-sequences of lenth 6 by Theorem 3.6, namely

- 1)  $(3,4,8,6,3,1) = (1,1,1,1,1,1)+(2,3,7,5,2,0)$ ,
- 2)  $(3,4,8,6,3,1) = (1,2,2,2,2,1)+(2,2,6,4,1,0)$ ,
- 3)  $(3,4,8,6,3,1) = (1,2,3,3,2,1)+(2,2,5,3,1,0)$ ,
- 4)  $(3,4,8,6,3,1) = (1,3,3,3,3,1)+(2,1,5,3,0,0)$ ,
- 5)  $(3,4,8,6,3,1) = (1,3,4,4,3,1)+(2,1,4,2,0,0)$ ,
- 6)  $(3,4,8,6,3,1) = (1,3,5,5,3,1)+(2,1,3,1,0,0)$ ,
- 7)  $(3,4,8,6,3,1) = (1,3,6,6,3,1)+(2,1,2,0,0,0)$ .

But, in 1),

$$\begin{aligned}
(2, 3, 7, 5, 2, 0) &= (1, 1, 1, \alpha, \beta, 0) + (1, 2, 6, 5 - \alpha, 2 - \beta, 0), \\
&\quad \alpha \leq 1 \\
&= (1, 2, 2, \alpha, \beta, 0) + (1, 1, 5, 5 - \alpha, 2 - \beta, 0), \\
&\quad \alpha \leq 2 \\
&= (1, 2, 3, \alpha, \beta, 0) + (1, 1, 4, 5 - \alpha, 2 - \beta, 0), \\
&\quad \alpha \leq 4 \\
&= (1, 3, \alpha, \beta, \gamma, 0) + (1, 0, 7 - \alpha, 5 - \beta, 2 - \gamma, 0), \\
&\quad \alpha \leq 6
\end{aligned}$$

can't be as a sum of two O-sequences since  $(1, 2, 6, 5 - \alpha, 2 - \beta, 0)$ ,  $(1, 1, 5, 5 - \alpha, 2 - \beta, 0)$ ,  $(1, 1, 4, 5 - \alpha, 2 - \beta, 0)$  and  $(1, 0, 7 - \alpha, 5 - \beta, 2 - \gamma, 0)$  are not O-sequences.

In 2),

$$\begin{aligned}
(2, 2, 6, 4, 1, 0) &= (1, 1, 1, \alpha, \beta, 0) + (1, 1, 5, 4 - \alpha, 1 - \beta, 0), \\
&\quad \alpha \leq 1 \\
&= (1, 2, \alpha, \beta, \gamma, 0) + (1, 0, 6 - \alpha, 4 - \beta, 1 - \gamma, 0), \\
&\quad \alpha \leq 3
\end{aligned}$$

cannot be as a sum of two O-sequences since  $(1, 1, 5, 4 - \alpha, 1 - \beta, 0)$  and  $(1, 0, 6 - \alpha, 4 - \beta, 1 - \gamma, 0)$  are not O-sequences.

In 3),

$$\begin{aligned}
(2, 2, 5, 3, 1, 0) &= (1, 1, \alpha, \beta, \gamma, 0) + (1, 1, 5 - \alpha, 3 - \beta, 1 - \gamma, 0), \alpha \leq 1 \\
&= (1, 2, \alpha, \beta, \gamma, 0) + (1, 0, 5 - \alpha, 3 - \beta, 1 - \gamma, 0), \alpha \leq 3
\end{aligned}$$

cannot be as a sum of two O-sequences since  $(1, 1, 5 - \alpha, 3 - \beta, 1 - \gamma, 0)$  and  $(1, 0, 5 - \alpha, 3 - \beta, 1 - \gamma, 0)$  are not O-sequences since  $5 - \alpha \geq 2$ .

In 4),  $(2, 1, 5, 3, 0, 0) = (1, 1, \alpha, \beta, 0, 0) + (1, 0, 5 - \alpha, 3 - \beta, 0, 0)$ ,  $\alpha \leq 1$  cannot be as a sum of two O-sequences since  $(1, 0, 5 - \alpha, 3 - \beta, 0, 0)$  is not O-sequence since  $5 - \alpha \geq 4$ .

By the same idea as above, we one can show that the cases 5), 6), and 7) cannot be as a sum of two O-sequences.

Therefore 54) 1 3 6 8 4 3 is not level, as we wished.  $\square$

Using the same idea as in the proof of Proposition 12, one can show that the following two more case cannot level;

27)	1, 3, 5, 7, 4, 3	61)	1, 3, 6, 9, 4, 3	68)	1, 3, 6, 10, 4, 3
69)	1, 3, 6, 10, 5, 3				

TABLE 4

**Definition 3.9.** (a) A total order on the monomials of each degree

is said to be a **term order** if

- i)  $x_1 > \cdots > x_n$ , and
- ii)  $m_1 \geq m_2$  implies  $mm_1 \geq mm_2$ , for any monomials  $m, m_1$  and  $m_2$ .

(b) The **reverse lexicographic order** is a term order defined to

be  $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$  if and only if

- i)  $\sum i_t > \sum j_t$  or
- ii)  $\sum i_t = \sum j_t$  and

there is  $s$  such that  $i_t = j_t$  for  $s < t \leq n$  and  $i_s < j_s$ .

(c) The **lexicographic order** is a term order defined to be  $x_1^{i_1} \cdots x_n^{i_n} >$

$x_1^{j_1} \cdots x_n^{j_n}$  if and only if

- i)  $\sum i_t > \sum j_t$  or

ii)  $\sum i_t = \sum j_t$  and

there is  $s$  such that  $i_t = j_t$  for  $t < s \leq n$  and  $i_s > j_s$ .

(d) Let  $S$  be a subset of all monomials in  $R_d$ .  $S$  is a **lex-segment**

if a monomial  $m$  of degree  $d$  is in  $S$ , then every monomial  $m'$  of degree  $d$  in  $R_d$  such that  $m' > m$  is in  $S$ .

(e) Let  $I = \bigoplus_{t \geq 0} I_t$  be a graded ideal of  $R$ . We say that  $I$  is a

**lex-segment ideal** if for every  $t \geq 0$ ,  $I_t$  is generated (as a vector space) by a lex-segment.

By the following Proposition 3.11, the case 22) 1 3 5 6 4 3 is not level.

**Proposition 3.10.** *There is no level Artinian algebra with the Hilbert function 1 3 5 6 4 3.*

*Proof.* Assume there exists a homogeneous ideal  $I$  of  $R = k[x, y, z]$  such that the minimal free resolution of  $R/I$  is  $\mathbf{H}$  and  $R/I$  is level.

Here is the Betti diagram for the lex-segment ideal with this Hilbert function:

total	1	9	14	6
0	1	-	-	-
1	-	1	-	-
2	-	1	1	-
3	-	3	5	2
4	-	1	2	1
5	-	3	6	3

Suppose that  $I$  has 3 generators in degree 6. Let  $J$  be the ideal generated by the components of  $I$  of degree  $\leq 5$ . Then the Hilbert function of  $R/J$  is  $1 \ 3 \ 5 \ 6 \ 4 \ 3 \ 3 \ \cdots$ . Note that the growth from degree 5 to degree 6 is maximal, and that in degrees 4, 5 and 6 the sequence  $3 \ 3 \ 3$  is a maximal sequence. Therefore, by Theorem 3.3,  $R/J$  has a socle element in degree 4. Hence also  $R/I$  has such a socle element. It follows that in order for  $R/I$  to be level,  $I$  must have at most 2 generators in degree 6. But then both copies  $R(-6)$  of the last free module cannot be canceled.  $\square$

By the same idea as above, one can prove that the case 42) is not level.

**Remark 3.11.** (Non-Cancellable Case) Let  $I$  be a lex-segment ideal in  $R$  with Hilbert function  $1 \ 3 \ 6 \ 6 \ 7 \ 3$ . Then the minimal free resolution of  $R/I$  is

$$\begin{aligned} 0 &\rightarrow R(-5) \oplus R^4(-7) \oplus R^3(-8) &\rightarrow R^4(-4) \oplus R^9(-6) \oplus R^6(-7) \\ &\rightarrow R^4(-3) \oplus R^5(-5) \oplus R^3(-6) &\rightarrow R &\rightarrow R/I &\rightarrow 0, \end{aligned}$$

and hence the copy  $R(-5)$  in the last free module is not cancelable, that is, the case 45)  $1 \ 3 \ 6 \ 6 \ 7 \ 3$  is not level. By the same idea, one can prove that the cases 47), 52) and 62) cannot be level.

By Example 3.13 in [4], the case 38)  $1 \ 3 \ 6 \ 5 \ 4 \ 3$  is not level, either.

#### 4. Some Construction of Gorenstein and Level Artinian O-sequences

**Definition 4.1** (Definition 3.1, [?]). (a) A finite set  $\mathbb{X}$  of points  $\mathbb{P}^2$  is called a **basic configuration** of type  $(d, e)$  if there exist distinct elements  $b_j, c_j$  in  $k$  such that

$$I_{\mathbb{X}} = \left( \prod_{j=1}^d (x - b_j z), \prod_{j=1}^e (y - c_j z) \right).$$

We denote  $\mathbb{X} := \mathbb{B}(d, e)$ .

(b) A finite set  $\mathbb{X}$  of points in  $\mathbb{P}^2$  is called a **pure configuration** if there exist finite basic configurations  $\mathbb{B}(d_1, e_1), \dots, \mathbb{B}(d_m, e_m)$  where  $e_1 > \dots > e_m$ , which satisfy the following three conditions:

- (i)  $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \emptyset$  if  $i \neq j$ ,
- (ii)  $\mathbb{X} = \mathbb{B}(d_1, e_1) \cup \dots \cup \mathbb{B}(d_m, e_m)$ ,
- (iii)  $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$  for all  $1 \leq i \leq m - 1$ , where  $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \rightarrow \mathbb{P}^1$  is the map defined by sending the point  $(x, y, z)$  to the point  $(y, z)$ . In this case, we denote  $\mathbb{X} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$ .

**Proposition 4.2** (Proposition 3.8, [8]). *Let  $\mathbb{X} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$  be a pure configuration in  $\mathbb{P}^2$ . Then a minimal free resolution of  $\mathbb{X}$  is :*

$$0 \rightarrow \bigoplus_{i=1}^m R(-p_i) \rightarrow \bigoplus_{i=1}^{m+1} R(-q_i) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0,$$

where

$$\begin{aligned} q_1 &= e_1, & q_i &= d_1 + \cdots + d_{i-1} + e_i \quad (2 \leq i \leq m), \\ q_{m+1} &= d_1 + \cdots + d_m, & p_i &= q_i + d_i \quad (1 \leq i \leq m). \end{aligned}$$

**Corollary 4.3** (Corollary 3.10, [8]). *Let  $\mathbb{X} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$  be a pure configuration in  $\mathbb{P}^2$ . Then  $\mathbb{X}$  is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all  $1 \leq i \leq m - 1$ .

If  $r(A)$  is Cohen-Macaulay type of a Cohen-Macaulay standard graded  $k$ -algebra  $A$ , and if  $A = \bigoplus_{i \geq 0} A_i$  is an Artinian level algebra, then  $r(A) = \dim_k A_{\sigma(A)-1}$ , where  $\sigma(A) = \min\{i | A_i = 0\}$ . If  $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$  is a level pure configuration in  $\mathbb{P}^2$ , then  $r(\mathbb{Z}) = r(R/I_{\mathbb{Z}}) = m$ .

**Lemma 4.4** (Lemma 3.14, [8]). *Let  $\mathbb{Z}$  be a level set of points in  $\mathbb{P}^n$  and  $\mathbb{X}$  a subset of  $\mathbb{Z}$ . Set  $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$ . Then  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  is an Artinian level graded  $k$ -algebra with  $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \sigma(\mathbb{Z}) - 1$  and  $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq r(\mathbb{Z})$ .*

**Corollary 4.5** (Corollary 3.15, [8]). *Let  $\mathbb{Z} = \bigcup_{i=1}^m \mathbb{B}(d_i, e_i)$  be a level pure configuration in  $\mathbb{P}^2$  and  $\mathbb{X}$  a subset of  $\mathbb{Z}$ . Set  $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$ . Then  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  is an Artinian level graded  $k$ -algebra with  $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = d_1 + e_1 - 2$  and  $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leq m$ .*

Now we classify all cases in Table 1 in the following example.

**Theorem 4.6** (Proposition 5.24, [4]). *Let  $(1, h_1, h_2, \dots, h_s)$  be the  $h$ -vector of a level algebra  $A = R/I$  where  $R = k[x_1, \dots, x_n]$ . Let  $j = \max\{i \mid h_i = \binom{i+n-1}{i}\}$ . Then, there is an ideal  $J \subset I$  so that  $B = R/J$  is a level algebra with  $h$ -vector  $(1, h_1, \dots, h_j, h_{j+1} + 1, \dots, h_s + 1)$ .*

Before we use Theorem 4.6, we shall show how to construct the following two level O-sequences  $(1, 3, 6, 7, 4, 2)$  and  $(1, 3, 6, 9, 5, 2)$  using link-sum construction.

**Example 4.7.** We start with a level set of points,  $\mathbb{Z}$  in  $\mathbb{P}^2$  which we partition into two subsets  $\mathbb{X}$  and  $\mathbb{Y}$ . The points of  $\mathbb{X}$  will be denoted with  $\bullet$ 's and those of  $\mathbb{Y}$  with  $*$ 's. Next to each diagram we give the Hilbert functions of all the relevant rings.

$$\mathbb{Z} = \begin{cases} * \\ * \\ * \\ \bullet & * & * & * \\ * & * & \bullet & \bullet \\ * & \bullet & \bullet & * \\ * & \bullet & \bullet & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 14 & 17 & 19 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 11 & 12 & 12 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 7 & 4 & 2 & 0 & \rightarrow . \end{aligned}$$

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & * \\ \bullet & \bullet & * & * \\ \bullet & * & * & * \\ * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 14 & 17 & 19 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 9 & 5 & 2 & 0 & \rightarrow . \end{aligned}$$

Now we can apply Theorem 4.6 to get level Artinian O-sequences for the following : 55) from  $(1, 3, 6, 7, 4, 2)$ , 70) from  $(1, 3, 6, 9, 5, 2)$ .

**Example 4.8.** We can construct 32 cases among 73 cases in Table 1, which are: 1), 7), 10), 11), 12), 18), 19), 23), 24), 25), 28), 29), 30), 31), 32), 43), 44), 48), 49), 50), 51), 56), 57), 58), 59), 63), 64), 65), 66), 71), 72), 73).

1)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 3 & 3 & 3 & 3 & 3 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 21 & 21 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 3 & 3 & 3 & 3 & 0 & \rightarrow . \end{aligned}$$

7)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & \bullet & * \\ * & \bullet & * & \bullet \\ * & * & * & * & * & * \\ * & * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 4 & 4 & 4 & 4 & 4 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 20 & 20 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 4 & 4 & 4 & 3 & 0 & \rightarrow . \end{aligned}$$

10)

$$\mathbb{Z} = \begin{cases} * \\ \bullet & \bullet \\ * & * & * \\ * & \bullet & * \\ * & \bullet & * \\ * & \bullet & * \\ * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 9 & 12 & 15 & 18 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 4 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 9 & 11 & 13 & 13 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 4 & 5 & 4 & 3 & 0 & \rightarrow . \end{aligned}$$

11)

$$\mathbb{Z} = \begin{cases} * & \bullet \\ * & \bullet \\ * & \bullet & * & * \\ * & * & * & * \\ * & \bullet & * & * & * & * \\ * & * & * & * & \bullet & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 4 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 19 & 19 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 4 & 5 & 5 & 3 & 0 & \rightarrow . \end{aligned}$$

12)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ \bullet & * \\ \bullet & * & \bullet & * \\ \bullet & * & * & * \\ * & * & * & * & * & * \\ \bullet & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 4 & 5 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 18 & 18 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 4 & 5 & 6 & 3 & 0 & \rightarrow . \end{aligned}$$

18)

$$\mathbb{Z} = \begin{cases} \bullet \\ * & * \\ * & * & * \\ * & * & \bullet \\ * & * & \bullet \\ * & \bullet & \bullet \\ * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 9 & 12 & 15 & 18 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 5 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 9 & 11 & 13 & 13 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 5 & 5 & 4 & 3 & 0 & \rightarrow . \end{aligned}$$

19)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ * & \bullet \\ \bullet & * & * & * \\ * & * & * & * \\ * & * & * & * & * & * \\ * & * & \bullet & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 5 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 19 & 19 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 5 & 5 & 5 & 3 & 0 & \rightarrow . \end{aligned}$$

23)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & \bullet \\ * & * & * & \bullet \\ * & * & * & * & * & * \\ * & \bullet & * & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 6 \ 6 \ 6 \ 6 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 18 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 6 \ 5 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

24)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & * \\ * & \bullet & * & * \\ \bullet & * & \bullet & \bullet & \bullet & * \\ * & * & \bullet & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 6 \ 6 \ 6 \ 6 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 18 \ 18 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 6 \ 6 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

25)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & \bullet \\ * & * & * & * \\ \bullet & * & \bullet & \bullet & \bullet & \bullet \\ * & * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 6 \ 7 \ 7 \ 7 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 17 \ 17 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 6 \ 7 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

28)

$$\mathbb{Z} = \begin{cases} \bullet \\ * & * \\ * & * & * \\ \bullet & * & \bullet \\ \bullet & * & \bullet \\ \bullet & * & \bullet \\ * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 9 & 12 & 15 & 18 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 5 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 9 & 10 & 11 & 11 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 5 & 7 & 5 & 3 & 0 & \rightarrow . \end{aligned}$$

29)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & \bullet & \bullet & \bullet & \bullet \\ * & \bullet & * & * & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 5 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 14 & 17 & 17 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 5 & 7 & 6 & 3 & 0 & \rightarrow . \end{aligned}$$

30)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & * \\ * & * & * & * \\ \bullet & \bullet & \bullet & * & * & \bullet \\ \bullet & \bullet & * & \bullet & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 7 \ 7 \ 7 \ 7 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 17 \ 17 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 7 \ 7 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

31)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ \bullet & * \\ * & * & * & * \\ \bullet & * & * & * \\ \bullet & \bullet & \bullet & \bullet & \bullet & * \\ * & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ 8 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 16 \ 16 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 7 \ 8 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

32)

$$\mathbb{Z} = \begin{cases} * & * \\ \bullet & * \\ \bullet & * & * & \bullet \\ \bullet & \bullet & \bullet & * \\ \bullet & * & \bullet & \bullet & * & \bullet \\ \bullet & * & \bullet & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 15 \ 15 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 5 \ 7 \ 9 \ 9 \ 9 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 5 \ 7 \ 9 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

43)

$$\mathbb{Z} = \begin{cases} * & \bullet \\ * & \bullet \\ * & \bullet & \bullet & \bullet \\ * & * & * & * \\ * & * & \bullet & * & * & * \\ * & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 14 & 18 & 18 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 6 & 5 & 3 & 0 & \rightarrow . \end{aligned}$$

44)

$$\mathbb{Z} = \begin{cases} * & * \\ \bullet & \bullet \\ \bullet & * & * & * \\ * & * & \bullet & * \\ * & * & * & * & * & * \\ * & * & * & \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 18 & 18 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 6 & 6 & 3 & 0 & \rightarrow . \end{aligned}$$

48)

$$\mathbb{Z} = \begin{cases} * \\ * & \bullet \\ * & \bullet & * \\ * & \bullet & \bullet \\ \bullet & * & * \\ * & * & \bullet \\ * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 9 & 12 & 15 & 18 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 9 & 10 & 11 & 11 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 7 & 5 & 3 & 0 & \rightarrow . \end{aligned}$$

49)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ * & * \\ \bullet & * & * & \bullet \\ \bullet & * & \bullet & \bullet \\ * & * & * & * & * & * \\ * & * & * & \bullet & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 14 & 17 & 17 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 7 & 6 & 3 & 0 & \rightarrow . \end{aligned}$$

50)

$$\mathbb{Z} = \begin{cases} \bullet & \bullet \\ * & \bullet \\ * & * & * & * \\ * & \bullet & * & * \\ * & * & * & * & * & * \\ * & * & \bullet & \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 17 & 17 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 7 & 7 & 3 & 0 & \rightarrow . \end{aligned}$$

51)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & * \\ \bullet & * & * & * \\ * & * & \bullet & * & * & \bullet \\ \bullet & \bullet & * & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 7 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 7 & 8 & 3 & 0 & \rightarrow . \end{aligned}$$

56)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & \bullet & \bullet \\ * & * & \bullet & \bullet \\ * & \bullet & \bullet & * & \bullet & \bullet \\ * & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 13 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 8 & 6 & 3 & 0 & \rightarrow . \end{aligned}$$

57)

$$\mathbb{Z} = \begin{cases} * & \bullet \\ * & \bullet \\ * & \bullet & \bullet & * \\ * & \bullet & * & \bullet \\ * & * & * & \bullet & * & * \\ * & * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 14 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 8 & 7 & 3 & 0 & \rightarrow . \end{aligned}$$

58)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ \bullet & * \\ * & * & * & * \\ * & * & * & * \\ \bullet & * & * & \bullet & \bullet & \bullet \\ * & * & \bullet & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 8 & 8 & 8 & 8 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 15 & 16 & 16 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 8 & 8 & 3 & 0 & \rightarrow . \end{aligned}$$

59)

$$\mathbb{Z} = \begin{cases} * & * \\ * & \bullet \\ * & * & \bullet & * \\ * & * & * & \bullet \\ * & * & * & * & \bullet & \bullet \\ \bullet & \bullet & * & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 8 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 14 & 14 & 14 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 8 & 9 & 3 & 0 & \rightarrow . \end{aligned}$$

63)

$$\mathbb{Z} = \begin{cases} \bullet \\ * & \bullet \\ * & \bullet & \bullet \\ * & * & \bullet \\ \bullet & * & * \\ * & \bullet & * \\ \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 9 & 12 & 15 & 18 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 9 & 6 & 3 & 0 & \rightarrow . \end{aligned}$$

64)

$$\mathbb{Z} = \begin{cases} \bullet & * \\ * & * \\ * & * & \bullet & \bullet \\ \bullet & * & \bullet & * \\ * & * & * & * & * & * \\ \bullet & * & * & \bullet & \bullet & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 13 \ 15 \ 15 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 9 \ 7 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

65)

$$\mathbb{Z} = \begin{cases} * & * \\ * & \bullet \\ \bullet & * & \bullet & \bullet \\ * & * & \bullet & * \\ * & * & \bullet & \bullet & * & \bullet \\ \bullet & * & * & * & * & * \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 14 \ 15 \ 15 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 9 \ 8 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

66)

$$\mathbb{Z} = \begin{cases} * & * \\ * & \bullet \\ * & \bullet & * & \bullet \\ \bullet & \bullet & * & \bullet \\ \bullet & * & * & * & * & * \\ \bullet & * & * & * & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \ \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 9 \ 9 \ 9 \ 9 \ \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 10 \ 15 \ 15 \ 15 \ \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 \ 3 \ 6 \ 9 \ 9 \ 3 \ 0 \ \rightarrow . \end{aligned}$$

71)

$$\mathbb{Z} = \begin{cases} * & * \\ * & * \\ * & * & * & \bullet \\ * & \bullet & \bullet & \bullet \\ * & * & \bullet & * & \bullet & \bullet \\ * & * & \bullet & \bullet & * & \bullet \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 12 & 14 & 14 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 10 & 7 & 3 & 0 & \rightarrow . \end{aligned}$$

72)

$$\mathbb{Z} = \begin{cases} * & * & & & & & & & \\ * & * & & & & & & & \\ * & * & * & \bullet & & & & & \\ * & \bullet & * & \bullet & & & & & \\ * & \bullet & \bullet & \bullet & * & * & & & \\ * & \bullet & \bullet & \bullet & * & \bullet & & & \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 13 & 14 & 14 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 10 & 8 & 3 & 0 & \rightarrow . \end{aligned}$$

73)

$$\mathbb{Z} = \begin{cases} \bullet & \bullet & & & & & & & \\ * & * & & & & & & & \\ \bullet & \bullet & * & \bullet & & & & & \\ * & * & \bullet & * & & & & & \\ \bullet & * & * & \bullet & \bullet & \bullet & & & \\ * & \bullet & \bullet & * & \bullet & \bullet & & & \end{cases}$$

Then the Hilbert functions of  $\mathbb{X}$ ,  $\mathbb{Y}$ ,  $\mathbb{Z}$ , and  $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$  are

$$\begin{aligned} \mathbf{H}_{\mathbb{Z}} &: 1 & 3 & 6 & 10 & 15 & 21 & 24 & \rightarrow \\ \mathbf{H}_{\mathbb{X}} &: 1 & 3 & 6 & 10 & 14 & 14 & 14 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}} &: 1 & 3 & 6 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} &: 1 & 3 & 6 & 10 & 9 & 3 & 0 & \rightarrow . \end{aligned}$$

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