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석사학위청구논문

ARTINIAN O-SEQUENCE OF SOCLE DEGREE 6 AND TYPE 3

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CONTENTS

논 문 개 요

Fröberg and Laksov 의 정리에 근거하여 여차원이 3, 형태가 3, 길이가 7인 334개의 Artinian O-수열을 CoCoA를 이용하여 찾았고 그 중 level이 되지 않는 경우에 대해 연구하였다.

Theorem과 Remark를 통해 level이 되지 않는 O-수열을 찾아 각 경우에 해당하는 Table을 만들었다. 또한, 앞의 방법을 통해 제거되지 않은 O-수열 중 Level이 되지 않는 특별한 경우를 증명하였다.

그 결과, 334개의 O-수열 중 221개가 level이 되지 않음을 증명하였다. 위의 방법으로는 level 여부를 알 수 없는 O-수열 10개가 남았다.

1. **Introduction**

A Gorenstein algebra has been much studied, so we are interested in level Artinian O-sequences. There have been many results on type 2 of level Artinian O-sequences of codimension 3 with length 7 (see [9], [14]), but not much on type 3 with length 7 since the type 3 Artinian O-sequences are more complicated.

Our main interest is to examine what kinds of Artinian O-sequences of type 3 can be level or not. We attempt to prove in two ways that Artinian O-sequences of codimension 3 and type 3 with length 7 cannot be level. One is for the cases which need long proofs and the other is what can be proved by using different ideas.

It seems to be wise to start with the definition of Hilbert function. If we let $R = k[x_0, x_1, \ldots, x_n] = \bigoplus_{i\geq 0} R_i$, where k is an algebraically closed field of characteristic 0, and let I be a homogeneous ideal of R, $A = R/I$, then the ***Hilbert function*** of A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R. The ***h-vector*** of A of $h(A) = (h_0, h_1, \ldots, h_\ell)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and ℓ is the last index such that $\dim_k A_k \neq 0$. We call ℓ the ***socle degree*** A. Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the ***codimension*** of A.

Using the algorithm in [13] from CoCoA [12] and a special case of a theorem of Fröberg and Laksov in [7], we have figured out that there are 334 Artinian O-sequences which could be level (See Appendix A). To investigate these cases further, we consider one of those sequences, 1 3 5 4 4 4 3. Let I be a lex-segment ideal of $R = k[x, y, z]$ with the Hilbert function 1 3 5 4 4. Then the minimal free resolution of R/I is

$$\begin{array}{llllll} 0 & \to & R^2(-5) \oplus R^4(-7) & \to & R^5(-4) \oplus R(-5) \oplus R^8(-6) & \\ & \to & R(-2) \oplus R^3(-3) \oplus R(-4) \oplus R^4(-5) & \to & R & \to \quad 0. \end{array}$$

This indicates that both copies $R(-5)$ of the last free module cannot be canceled. This means an Artinian algebra with Hilbert function having a sequence, 1 3 5 4 4, cannot be level, and thus, by Theorem 8 in [13], any Artinian algebra with Hilbert function having a sequence, 1 3 5 4 4 4 3, cannot be level, either.

We try to classify Artinian O-sequences of codimension 3 and type 3 with length 7 based on whether or not being level. In this thesis, we shall prove that some O-sequences among the above 334 cases, which look like possible level Artinian O-sequences, cannot be level Artinian O-sequences.

2. **Non-Level Artinian O-sequences**

of codimension 3 and type 3

First of all, we introduce some definitions here.

Definition 2.1. (a) Let $h,\ i > 0$.

Then $h = \binom{m_i}{i} + \binom{m_i - 1}{i-1} + \cdots + \binom{m_j}{j}$ with $m_i > m_i - 1 > \cdots > m_j \geq 1$.

We are called ***i-th binomial expansion***.

(b) The sequnce $\{h_i\}_{i\geq 0}$ $(h_i \geq 0)$ is called an ***O-Sequence*** if there is $h_{i+1} \leq h_i^{\langle i \rangle},\ h_0 = 1,\ i \geq 1$.

Definition 2.2. Let $A = \bigoplus_{i\geq 0} A_i$be graded ring. Then $A = k[A_1]$ is called ***G-algebra*** if A_1 is a finitely generated vector space over k.

Theorem 2.3 (Macaulay)**.** (a) $\{h_i\}_{i\geq 0}$ *is an O-Sequence.*

(b) $\{h_i\}_{i\geq 0}$ *is the Hilbert function for some standard G-algebra.*

Definition-Proposition 2.4 (Definition-Proposition 2.21 in [10])**.** Let $R = k[x_0, \dots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. Then

(a) If $B = B_0 \oplus \cdots \oplus B_\ell$ ($B_\ell \neq 0$) is an Artinian algebra, then B is ***level*** if and only if $B_\ell = \mathrm{Ann}(B_1)$.

(b) A is a ***level algebra*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. rank $\mathcal{F}_{n-(d-1)}$
$=$ Cohen-Macaulay type of A.

(c) i) If $\mathbb{X}$ is a non-degenerate set of points in $\mathbb{P}^n$, $A = R/I_{\mathbb{X}}$ its coordinate ring, then we say that ℓ is the ***socle degree*** of $\mathbb{X}$ if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where $\overline{L}$ is any linear non-zero-divisor of A.

ii) $\mathbb{X}$ is called a ***level set*** of points if $A = R/I_{\mathbb{X}}$ is a level algebra. In this case, the socle degree of $\mathbb{X}$ is $\ell = \sigma(\mathbb{X}) + n - 1$.

(d) If $\overline{L}$ is a linear non-zero divisor in $A = R/I$, then A is level if and only if $A/\overline{L}A \simeq A/(L, I_{\mathbb{X}})$ is level.

(e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an

Artinian algebra) $b = \{b_i\}_{i\geq 0}$ with $b_1 = n+1$, is called ***level*** if there is a level set of points in with Hilbert function b.

Definition 2.5. (a) A total order on the monomials of each degree is said to be a ***term order*** if

i) $x_1 > \cdots > x_n$, and

ii) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m, m_1 and m_2.

(b) The ***reverse lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $s < t \leq n$ and $i_s < j_s$.

(c) The ***lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.

(d) Let S be a subset of all monomials in R_d. S is a ***lex-segment*** if a monomial m of deree d is in S, then every monomial m' of degree d in R_d such that $m' > m$ is in S.

(e) Let $I = \bigoplus_{t\geq 0} I_t$ be a graded ideal of R. We say that I is a ***lex-segment ideal*** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

Theorem 2.6 (The Cancelation Principle, [1], [11])**.** *For any homogeneous ideal I and any i and d, there a complex of $k \cong R/m-modules$ $V_\bullet^d$ such that*

$$\begin{array}{lcl} V_i^d & \cong & \operatorname{Tor}_i^R(\operatorname{in}(I), k)_d \\ H_i(V_\bullet^d) & \cong & \operatorname{Tor}_i^R(I, k)_d. \end{array}$$

Remark 2.7. One way to paraphrase this theorem is to say that the minimal free resolution of I is obtained from that $\operatorname{in}(I)$, the *initial ideal* of I, by canceling some adjacent terms of the same degree.

The following cases in Table 1 are not level by Theorem 2.6 and Remark 2.7.

62)	1 3 5 6 7 5 3	78)	1 3 5 7 6 7 3	86)	1 3 5 7 8 4 3
93)	1 3 5 7 9 4 3	133)	1 3 6 6 7 5 3	134)	1 3 6 6 7 6 3
135)	1 3 6 6 7 7 3	136)	1 3 6 6 7 8 3	164)	1 3 6 7 9 5 3
165)	1 3 6 7 9 6 3	166)	1 3 6 7 9 7 3	167)	1 3 6 7 9 8 3
168)	1 3 6 7 9 9 3	173)	1 3 6 8 5 4 3	193)	1 3 6 8 8 9 3
202)	1 3 6 8 10 4 3	203)	1 3 6 8 10 5 3	212)	1 3 6 9 5 4 3
217)	1 3 6 9 6 5 3	232)	1 3 6 9 8 9 3	242)	1 3 6 9 10 5 3

249)	1 3 6 9 11 5 3	255)	1 3 6 9 12 4 3	256)	1 3 6 9 12 5 3
265)	1 3 6 10 5 4 3	270)	1 3 6 10 6 5 3	272)	1 3 6 10 6 7 3
278)	1 3 6 10 7 8 3	285)	1 3 6 10 8 9 3	295)	1 3 6 10 10 5 3
302)	1 3 6 10 11 5 3	309)	1 3 6 10 12 5 3	315)	1 3 6 10 13 4 3
316)	1 3 6 10 13 5 3	317)	1 3 6 10 13 6 3	322)	1 3 6 10 14 4 3
323)	1 3 6 10 14 5 3	324)	1 3 6 10 14 6 3	329)	1 3 6 10 15 4 3
330)	1 3 6 10 15 5 3	331)	1 3 6 10 13 6 3		

TABLE 1

Theorem 2.8 ([2]). *Let* $R = k[x_1, x_2, x_3]$ *and let* $\mathbf{H} = (h_0, h_1, \dots, h_s)$ *be the h-vector of a graded Artinian algebra* $A = R/I$ *with socle degree* s. *If*

$$h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \leq 2d + 2$$

then $\mathbf{H}$ *is* **not** *level.*

Remark 2.9. Let $\mathbf{H}$ and R be as above. Then any Artinian algebra $A = R/I$ with Hilbert function $\mathbf{H}$ has a socle element in degree $d - 1$.

The following cases in Table 2 are not level by Theorem 2.8.

2)	1 3 3 4 3 3 3	3)	1 3 3 4 4 3 3	5)	1 3 3 4 5 3 3
9)	1 3 4 3 3 3 3	10)	1 3 4 4 3 3 3	11)	1 3 4 4 4 3 3
13)	1 3 4 4 5 3 3	17)	1 3 4 5 3 3 3	19)	1 3 4 5 4 4 3
20)	1 3 4 5 5 3 3	24)	1 3 4 5 6 3 3	29)	1 3 5 3 3 3 3

30)	1 3 5 4 3 3 3	31)	1 3 5 4 4 3 3	32)	1 3 5 4 4 4 3
37)	1 3 5 5 3 3 3	38)	1 3 5 5 4 3 3	39)	1 3 5 5 4 4 3
40)	1 3 5 5 5 3 3	44)	1 3 5 5 6 3 3	49)	1 3 5 6 3 3 3
51)	1 3 5 6 4 4 3	54)	1 3 5 6 5 5 3	56)	1 3 5 6 6 3 3
61)	1 3 5 6 7 3 3	67)	1 3 5 7 3 3 3	69)	1 3 5 7 4 4 3
72)	1 3 5 7 5 5 3	77)	1 3 5 7 6 6 3	79)	1 3 5 7 7 3 3
85)	1 3 5 7 8 3 3	92)	1 3 5 7 9 3 3	99)	1 3 6 3 3 3 3
101)	1 3 6 4 4 3 3	102)	1 3 6 4 4 4 3	109)	1 3 6 5 4 4 3
110)	1 3 6 5 5 3 3	111)	1 3 6 5 5 4 3	112)	1 3 6 5 5 5 3
113)	1 3 6 5 5 6 3	119)	1 3 6 6 3 3 3	120)	1 3 6 6 4 3 3
121)	1 3 6 6 4 4 3	122)	1 3 6 6 5 3 3	124)	1 3 6 6 5 5 3
126)	1 3 6 6 6 3 3	131)	1 3 6 6 7 3 3	137)	1 3 6 7 3 3 3
139)	1 3 6 7 4 4 3	142)	1 3 6 7 5 5 3	147)	1 3 6 7 6 6 3
149)	1 3 6 7 7 3 3	155)	1 3 6 7 8 3 3	162)	1 3 6 7 9 3 3
169)	1 3 6 8 3 3 3	171)	1 3 6 8 4 4 3	174)	1 3 6 8 5 5 3
179)	1 3 6 8 6 6 3	185)	1 3 6 8 7 7 3	187)	1 3 6 8 8 3 3
194)	1 3 6 8 9 3 3	201)	1 3 6 8 10 3 3	208)	1 3 6 9 3 3 3
210)	1 3 6 9 4 4 3	213)	1 3 6 9 5 5 3	218)	1 3 6 9 6 6 3
224)	1 3 6 9 7 7 3	231)	1 3 6 9 8 8 3	233)	1 3 6 9 9 3 3
240)	1 3 6 9 10 3 3	247)	1 3 6 9 11 3 3	254)	1 3 6 9 12 3 3
261)	1 3 6 10 3 3 3	263)	1 3 6 10 4 4 3	266)	1 3 6 10 5 5 3
271)	1 3 6 10 6 6 3	277)	1 3 6 10 7 7 3	284)	1 3 6 10 8 8 3
292)	1 3 6 10 9 9 3	293)	1 3 6 10 10 3 3	300)	1 3 6 10 11 3 3
307)	1 3 6 10 12 3 3	314)	1 3 6 10 13 3 3	321)	1 3 6 10 14 3 3
328)	1 3 6 10 15 3 3				

Table 2

Theorem 2.10 (Theorem 2.17, [9]). *Let $h_{d-2},\ h_{d-1},\ h_d$ be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d-1\rangle} \qquad \textit{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2\rangle}\ .$$

Let I be any ideal in $R = k[x_1, \ldots, x_n]$ such that the Hilbert function of R/I satisfies

$$\begin{array}{lcl} \mathbf{H}(R/I, d-2) & = & h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) & = & h_{d-1}, \\ \mathbf{H}(R/I, d) & = & h_d \, . \end{array}$$

Then, the ring R/I has socle of dimension ε in degree $d-2$.

Using Theorem 2.10, one can show that the following h-vectors in Table 3 are not level O-sequences.

4)	1 3 3 4 4 4 3	6)	1 3 3 4 5 4 3	7)	1 3 3 4 5 5 3
8)	1 3 3 4 5 6 3	14)	1 3 4 4 5 4 3	15)	1 3 4 4 5 5 3
16)	1 3 4 4 5 6 3	18)	1 3 4 5 4 3 3	23)	1 3 4 5 5 6 3
33)	1 3 5 4 5 3 3	34)	1 3 5 4 5 4 3	35)	1 3 5 4 5 5 3
36)	1 3 5 4 5 6 3	43)	1 3 5 5 5 6 3	45)	1 3 5 5 6 4 3
46)	1 3 5 5 6 5 3	47)	1 3 5 5 6 6 3	48)	1 3 5 5 6 7 3
50)	1 3 5 6 4 3 3	52)	1 3 5 6 5 3 3	55)	1 3 5 6 5 6 3
60)	1 3 5 6 6 7 3	68)	1 3 5 7 4 3 3	70)	1 3 5 7 5 3 3
73)	1 3 5 7 5 6 3	74)	1 3 5 7 6 3 3	84)	1 3 5 7 7 8 3
100)	1 3 6 4 3 3 3	103)	1 3 6 4 5 3 3	104)	1 3 6 4 5 4 3
105)	1 3 6 4 5 5 3	106)	1 3 6 4 5 6 3	107)	1 3 6 5 3 3 3
108)	1 3 6 5 4 3 3	114)	1 3 6 5 6 3 3	115)	1 3 6 5 6 4 3
116)	1 3 6 5 6 5 3	117)	1 3 6 5 6 6 3	118)	1 3 6 5 6 7 3
125)	1 3 6 6 5 6 3	130)	1 3 6 6 6 7 3	138)	1 3 6 7 4 3 3
140)	1 3 6 7 5 3 3	143)	1 3 6 7 5 6 3	144)	1 3 6 7 6 3 3
148)	1 3 6 7 6 7 3	154)	1 3 6 7 7 8 3	170)	1 3 6 8 4 3 3
172)	1 3 6 8 5 3 3	175)	1 3 6 8 5 6 3	176)	1 3 6 8 6 3 3
180)	1 3 6 8 6 7 3	181)	1 3 6 8 7 3 3	186)	1 3 6 8 7 8 3
209)	1 3 6 9 4 3 3	211)	1 3 6 9 5 3 3	214)	1 3 6 9 5 6 3

215)	1 3 6 9 6 3 3	219)	1 3 6 9 6 7 3	220)	1 3 6 9 7 3 3
223)	1 3 6 9 7 6 3	225)	1 3 6 9 7 8 3	226)	1 3 6 9 8 3 3
262)	1 3 6 10 4 3 3	264)	1 3 6 10 5 3 3	267)	1 3 6 10 5 6 3
268)	1 3 6 10 6 3 3	273)	1 3 6 10 7 3 3	279)	1 3 6 10 8 3 3
286)	1 3 6 10 9 3 3				

Table 3

Proposition 2.11. *The Artinian O-sequence* $\mathbf{H}: 1 \quad 3 \quad 6 \quad 7 \quad 5 \quad 4 \quad 3$ *is not level.*

Proof. Let I be a lex-segment ideal in $R = k[x,y,z]$ with Hilbert function $\mathbf{H}: 1 \quad 3 \quad 6 \quad 7 \quad 5 \quad 4 \quad 3$. Then the minimal free resolution of R/I is

$$\begin{array}{rl} 0 & \to R(-5) \oplus R^3(-6) \oplus R(-7) \oplus R(-8) \oplus R^3(-9) \\ & \to R^3(-4) \oplus R^7(-5) \oplus R^3(-6) \oplus R^2(-7) \oplus R^6(-8) \\ & \to R^3(-3) \oplus R^4(-4) \oplus R^2(-5) \oplus R(-6) \oplus R^3(-7) \\ & \to R \to R/I \to 0. \end{array}$$

Suppose K is a homogeneous level ideal of R such that the Hilbert function of R/K is $\mathbf{H}$, and K has one generator in dgree 6.
Let J be the ideal generated by the components of K of degree ≤ 5.
Then the Hilbert function of R/J is $1 \quad 3 \quad 6 \quad 7 \quad 5 \quad 4 \quad 4 \cdots$.
Note that the growth from dgree 5 to dgree 6 is maximal, and that in degrees 4, 5 and 6, the sequence 4, 4, 4 is a maximal growth.
Therefore, by Theorem 2.10, R/J has a socle element in dgree 4. Hence also R/K has such a socle element. It follows that in order for R/K to be level K must have no generator in degree 6. But then three copies $R(-6)$ of the last free module cannot be canceled. □

Proposition 2.12. *The Artinian O-sequence* $\mathbf{H}: 1 \quad 3 \quad 6 \quad 8 \quad 6 \quad 5 \quad 3$ *is not level.*

Proof. Let I be a lex-segment ideal in $R = k[x, y, z]$ with Hilbert function $\mathbf{H}: 1 \quad 3 \quad 6 \quad 8 \quad 6 \quad 5 \quad 3$. Then the minimal free resolution of R/I is

$$\begin{array}{rl} 0 & \rightarrow & R^3(-6) \oplus R(-7) \oplus R^2(-8) \oplus R^3(-9) \\ & \rightarrow & R(-4) \oplus R^7(-5) \oplus R^3(-6) \oplus R^4(-7) \\ & \rightarrow & R^2(-3) \oplus R^4(-4) \oplus R^2(-5) \oplus R^2(-6) \oplus R^3(-7) \\ & \rightarrow & R \quad \rightarrow \quad R/I \quad \rightarrow \quad 0. \end{array}$$

Suppose K is a homogeneous level ideal of R such that the Hilbert function of R/K is $\mathbf{H}$, and K has two generator in dgree 6.
Let J be the ideal generated by the components of K of degree ≤ 5.
Then the Hilbert function of R/J is $1 \quad 3 \quad 6 \quad 8 \quad 6 \quad 5 \quad 5 \cdots$.
Then, by Remark 2.9, R/J has a socle element in degree 4.
Hence also R/K has such a socle element. It follows that in order for R/K to be level K must have at most one generator in degree 6. But then three copies $R(-6)$ of the last free module cannot be canceled. $\square$

By the similar argument as above, we can show that the following cases in Table 4 are not level.

22)	1 3 4 5 5 5 3	42)	1 3 5 5 5 5 3	71)	1 3 5 7 5 4 3
141)	1 3 6 7 5 4 3	178)	1 3 6 8 6 5 3	269)	1 3 6 10 6 4 3
276)	1 3 6 10 7 6 3				

TABLE 4

Proposition 2.13 ([9]). *Let h_{d-2}, h_{d-1}, h_d be three integers such that*

$$h_d = h_{d-1}^{\langle d-1\rangle} \qquad \text{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2\rangle} \ .$$

Let I be any ideal in $R = k[x_1, \ldots, x_n]$ for which

$$\begin{array}{lcl} \mathbf{H}(R/I, d-2) & = & h_{d-2} + \varepsilon, \quad \varepsilon \geq n, \\ \mathbf{H}(R/I, d-1) & = & h_{d-1}, \\ \mathbf{H}(R/I, d) & = & h_d - 1\,. \end{array}$$

Then $\mathtt{dim}_k soc(R/I)_{d-2} \geq 1$ and so any O-sequence

$$(1, n, \ldots, h_{d-2} + \varepsilon, h_{d-1}, h_d - 1, \ldots, h_s)$$

of length $s+1$ ($s \geq d$) is not a level O-sequence.

The following cases in Talbe 5 are not level by Proposition 2.13.

80)	1 3 5 7 7 4 3	132)	1 3 6 6 7 4 3	150)	1 3 6 7 7 4 3
156)	1 3 6 7 8 4 3	163)	1 3 6 7 9 4 3	182)	1 3 6 8 7 4 3
188)	1 3 6 8 8 4 3	195)	1 3 6 8 9 4 3	221)	1 3 6 9 7 4 3
227)	1 3 6 9 8 4 3	234)	1 3 6 9 9 4 3	241)	1 3 6 9 10 4 3
248)	1 3 6 9 11 4 3	274)	1 3 6 10 7 4 3	280)	1 3 6 10 8 4 3

287)	1 3 6 10 9 4 3	294)	1 3 6 10 10 4 3	301)	1 3 6 10 11 4 3

TABLE 5

Proposition 2.14. *The Artinian O-sequence* 1 3 6 10 8 7 3 *is not level.*

Proof. Let I be a lex-segment ideal in $R = k[x, y, z]$ with Hilbert function $\mathbf{H}: 1 \quad 3 \quad 6 \quad 10 \quad 8 \quad 7 \quad 3$.

Then the minimal free resolution of R/I is

$$\begin{array}{rl} 0 & \to R^3(-6) \oplus R^2(-7) \oplus R^4(-8) \oplus R^3(-9) \\ & \to R^9(-5) \oplus R^4(-6) \oplus R^9(-7) \oplus R^6(-8) \\ & \to R^7(-4) \oplus R^2(-5) \oplus R^5(-6) \oplus R^3(-7) \\ & \to R \to R/I \to 0. \end{array}$$

If there were a level algebra with this h-vector we'd have to be able to cancel three copies $R(-6)$ of the last free module.

So, suppose that $A = R/I$ is a level algebra with Hilbert function $\mathbf{H}: 1 \quad 3 \quad 6 \quad 10 \quad 8 \quad 7 \quad 3$. If I has at most three generators in degree 6 then we'd have a contradiction : for then middle free module has at most two copies $R(-6)$. But, we need at least three copies $R(-6)$ of the middle free module to cancel three copies $R(-6)$ of the last free module. So, it suffices to show that I has at most three generators in degree 6 (we know that it has at most 5 generators in degree 6). Let $J = (I_{\leq 5})$.

Case 1. Suppose I has 5 generators in degree 6. Then the Hilbert function of R/J begins 1 3 6 10 8 7 8 $\cdots$. By Theorem 2.10,

R/J has a socle element in degree 4 and hence so does R/I, which is a contradiction.

Case 2. Now suppose that I has 4 generators in degree 6. Then the Hilbert function of $B = R/J$ begins $1 \quad 3 \quad 6 \quad 10 \quad 8 \quad 7 \quad 7 \quad t \quad \cdots$. We don't know exactly what t is but we can say that $0 \leq t \leq 7$. Then by Theorem 2.8, R/J has a socle element in degree 4 and so does R/I, which is a contradiction.

There I has at most three generators, and hence any Artinian algebra R/I with Hilbert function $\mathbf{H}$ cannot be level as we wished.

$\square$

Question 2.15. We have a few more cases similar to $\mathbf{H}$ in Proposition 2.15. For example, if I the lex-segment ideal of R with Hilbert function $\mathbf{H} : 1 \quad 3 \quad 6 \quad 10 \quad 7 \quad 5 \quad 3$, then R/I has a minimal free resolution as follows.

$$\begin{array}{rl} 0 \rightarrow & R^4(-6) \oplus R^2(-7) \oplus R^2(-8) \oplus R^3(-9) \\ \rightarrow & R^{11}(-5) \oplus R^5(-6) \oplus R^4(-7) \oplus R^6(-8) \\ \rightarrow & R^8(-4) \oplus R^3(-5) \oplus R^2(-6) \oplus R^3(-7) \\ \rightarrow & R \rightarrow R/I \rightarrow 0. \end{array}$$

Then the difference between the Betti number of the shift 6 in the last free module and the Betti number of the same shift in the middle free module is one as in the case of Proposition 2.14. However, we cannot say if they are level using the same idea as the proof in Proposition 2.14.

53)	1 3 5 6 5 4 3	76)	1 3 5 7 6 5 3	83)	1 3 5 7 7 7 3
123)	1 3 6 6 5 4 3	129)	1 3 6 6 6 6 3	146)	1 3 6 7 6 5 3
153)	1 3 6 7 7 7 3	192)	1 3 6 8 8 8 3	216)	1 3 6 9 6 4 3
239)	1 3 6 9 9 9 3				

Table 6

Appendix A

The total 334 Artinian cases based on Theorem of Fröberg and Laksov in [7].

1)	1 3 3 3 3 3 3	2)	1 3 3 4 3 3 3	3)	1 3 3 4 4 3 3
4)	1 3 3 4 4 4 3	5)	1 3 3 4 5 3 3	6)	1 3 3 4 5 4 3
7)	1 3 3 4 5 5 3	8)	1 3 3 4 5 6 3	9)	1 3 4 3 3 3 3
10)	1 3 4 4 3 3 3	11)	1 3 4 4 4 3 3	12)	1 3 4 4 4 4 3
13)	1 3 4 4 5 3 3	14)	1 3 4 4 5 4 3	15)	1 3 4 4 5 5 3
16)	1 3 4 4 5 6 3	17)	1 3 4 5 3 3 3	18)	1 3 4 5 4 3 3
19)	1 3 4 5 4 4 3	20)	1 3 4 5 5 3 3	21)	1 3 4 5 5 4 3
22)	1 3 4 5 5 5 3	23)	1 3 4 5 5 6 3	24)	1 3 4 5 6 3 3
25)	1 3 4 5 6 4 3	26)	1 3 4 5 6 5 3	27)	1 3 4 5 6 6 3
28)	1 3 4 5 6 7 3	29)	1 3 5 3 3 3 3	30)	1 3 5 4 3 3 3
31)	1 3 5 4 4 3 3	32)	1 3 5 4 4 4 3	33)	1 3 5 4 5 3 3
34)	1 3 5 4 5 4 3	35)	1 3 5 4 5 5 3	36)	1 3 5 4 5 6 3
37)	1 3 5 5 3 3 3	38)	1 3 5 5 4 3 3	39)	1 3 5 5 4 4 3
40)	1 3 5 5 5 3 3	41)	1 3 5 5 5 4 3	42)	1 3 5 5 5 5 3
43)	1 3 5 5 5 6 3	44)	1 3 5 5 6 3 3	45)	1 3 5 5 6 4 3
46)	1 3 5 5 6 5 3	47)	1 3 5 5 6 6 3	48)	1 3 5 5 6 7 3
49)	1 3 5 6 3 3 3	50)	1 3 5 6 4 3 3	51)	1 3 5 6 4 4 3
52)	1 3 5 6 5 3 3	53)	1 3 5 6 5 4 3	54)	1 3 5 6 5 5 3
55)	1 3 5 6 5 6 3	56)	1 3 5 6 6 3 3	57)	1 3 5 6 6 4 3
58)	1 3 5 6 6 5 3	59)	1 3 5 6 6 6 3	60)	1 3 5 6 6 7 3
61)	1 3 5 6 7 3 3	62)	1 3 5 6 7 4 3	63)	1 3 5 6 7 5 3
64)	1 3 5 6 7 6 3	65)	1 3 5 6 7 7 3	66)	1 3 5 6 7 8 3
67)	1 3 5 7 3 3 3	68)	1 3 5 7 4 3 3	69)	1 3 5 7 4 4 3
70)	1 3 5 7 5 3 3	71)	1 3 5 7 5 4 3	72)	1 3 5 7 5 5 3
73)	1 3 5 7 5 6 3	74)	1 3 5 7 6 3 3	75)	1 3 5 7 6 4 3
76)	1 3 5 7 6 5 3	77)	1 3 5 7 6 6 3	78)	1 3 5 7 6 7 3
79)	1 3 5 7 7 3 3	80)	1 3 5 7 7 4 3	81)	1 3 5 7 7 5 3
82)	1 3 5 7 7 6 3	83)	1 3 5 7 7 7 3	84)	1 3 5 7 7 8 3
85)	1 3 5 7 8 3 3	86)	1 3 5 7 8 4 3	87)	1 3 5 7 8 5 3
88)	1 3 5 7 8 6 3	89)	1 3 5 7 8 7 3	90)	1 3 5 7 8 8 3
91)	1 3 5 7 8 9 3	92)	1 3 5 7 9 3 3	93)	1 3 5 7 9 4 3

94)	1 3 5 7 9 5 3	95)	1 3 5 7 9 6 3	96)	1 3 5 7 9 7 3
97)	1 3 5 7 9 8 3	98)	1 3 5 7 9 9 3	99)	1 3 6 3 3 3 3
100)	1 3 6 4 3 3 3	101)	1 3 6 4 4 3 3	102)	1 3 6 4 4 4 3
103)	1 3 6 4 5 3 3	104)	1 3 6 4 5 4 3	105)	1 3 6 4 5 5 3
106)	1 3 6 4 5 6 3	107)	1 3 6 5 3 3 3	108)	1 3 6 5 4 3 3
109)	1 3 6 5 4 4 3	110)	1 3 6 5 5 3 3	111)	1 3 6 5 5 4 3
112)	1 3 6 5 5 5 3	113)	1 3 6 5 5 6 3	114)	1 3 6 5 6 3 3
115)	1 3 6 5 6 4 3	116)	1 3 6 5 6 5 3	117)	1 3 6 5 6 6 3
118)	1 3 6 5 6 7 3	119)	1 3 6 6 3 3 3	120)	1 3 6 6 4 3 3
121)	1 3 6 6 4 4 3	122)	1 3 6 6 5 3 3	123)	1 3 6 6 5 4 3
124)	1 3 6 6 5 5 3	125)	1 3 6 6 5 6 3	126)	1 3 6 6 6 3 3
127)	1 3 6 6 6 4 3	128)	1 3 6 6 6 5 3	129)	1 3 6 6 6 6 3
130)	1 3 6 6 6 7 3	131)	1 3 6 6 7 3 3	132)	1 3 6 6 7 4 3
133)	1 3 6 6 7 5 3	134)	1 3 6 6 7 6 3	135)	1 3 6 6 7 7 3
136)	1 3 6 6 7 8 3	137)	1 3 6 7 3 3 3	138)	1 3 6 7 4 3 3
139)	1 3 6 7 4 4 3	140)	1 3 6 7 5 3 3	141)	1 3 6 7 5 4 3
142)	1 3 6 7 5 5 3	143)	1 3 6 7 5 6 3	144)	1 3 6 7 6 3 3
145)	1 3 6 7 6 4 3	146)	1 3 6 7 6 5 3	147)	1 3 6 7 6 6 3
148)	1 3 6 7 6 7 3	149)	1 3 6 7 7 3 3	150)	1 3 6 7 7 4 3
151)	1 3 6 7 7 5 3	152)	1 3 6 7 7 6 3	153)	1 3 6 7 7 7 3
154)	1 3 6 7 7 8 3	155)	1 3 6 7 8 3 3	156)	1 3 6 7 8 4 3
157)	1 3 6 7 8 5 3	158)	1 3 6 7 8 6 3	159)	1 3 6 7 8 7 3
160)	1 3 6 7 8 8 3	161)	1 3 6 7 8 9 3	162)	1 3 6 7 9 3 3
163)	1 3 6 7 9 4 3	164)	1 3 6 7 9 5 3	165)	1 3 6 7 9 6 3
166)	1 3 6 7 9 7 3	167)	1 3 6 7 9 8 3	168)	1 3 6 7 9 9 3
169)	1 3 6 8 3 3 3	170)	1 3 6 8 4 3 3	171)	1 3 6 8 4 4 3
172)	1 3 6 8 5 3 3	173)	1 3 6 8 5 4 3	174)	1 3 6 8 5 5 3
175)	1 3 6 8 5 6 3	176)	1 3 6 8 6 3 3	177)	1 3 6 8 6 4 3
178)	1 3 6 8 6 5 3	179)	1 3 6 8 6 6 3	180)	1 3 6 8 6 7 3
181)	1 3 6 8 7 3 3	182)	1 3 6 8 7 4 3	183)	1 3 6 8 7 5 3
184)	1 3 6 8 7 6 3	185)	1 3 6 8 7 7 3	186)	1 3 6 8 7 8 3
187)	1 3 6 8 8 3 3	188)	1 3 6 8 8 4 3	189)	1 3 6 8 8 5 3
190)	1 3 6 8 8 6 3	191)	1 3 6 8 8 7 3	192)	1 3 6 8 8 8 3
193)	1 3 6 8 8 9 3	194)	1 3 6 8 9 3 3	195)	1 3 6 8 9 4 3
196)	1 3 6 8 9 5 3	197)	1 3 6 8 9 6 3	198)	1 3 6 8 9 7 3
199)	1 3 6 8 9 8 3	200)	1 3 6 8 9 9 3	201)	1 3 6 8 10 3 3

202)	1 3 6 8 10 4 3	203)	1 3 6 8 10 5 3	204)	1 3 6 8 10 6 3
205)	1 3 6 8 10 7 3	206)	1 3 6 8 10 8 3	207)	1 3 6 8 10 9 3
208)	1 3 6 9 3 3 3	209)	1 3 6 9 4 3 3	210)	1 3 6 9 4 4 3
211)	1 3 6 9 5 3 3	212)	1 3 6 9 5 4 3	213)	1 3 6 9 5 5 3
214)	1 3 6 9 5 6 3	215)	1 3 6 9 6 3 3	216)	1 3 6 9 6 4 3
217)	1 3 6 9 6 5 3	218)	1 3 6 9 6 6 3	219)	1 3 6 9 6 7 3
220)	1 3 6 9 7 3 3	221)	1 3 6 9 7 4 3	222)	1 3 6 9 7 5 3
223)	1 3 6 9 7 6 3	224)	1 3 6 9 7 7 3	225)	1 3 6 9 7 8 3
226)	1 3 6 9 8 3 3	227)	1 3 6 9 8 4 3	228)	1 3 6 9 8 5 3
229)	1 3 6 9 8 6 3	230)	1 3 6 9 8 7 3	231)	1 3 6 9 8 8 3
232)	1 3 6 9 8 9 3	233)	1 3 6 9 9 3 3	234)	1 3 6 9 9 4 3
235)	1 3 6 9 9 5 3	236)	1 3 6 9 9 6 3	237)	1 3 6 9 9 7 3
238)	1 3 6 9 9 8 3	239)	1 3 6 9 9 9 3	240)	1 3 6 9 10 3 3
241)	1 3 6 9 10 4 3	242)	1 3 6 9 10 5 3	243)	1 3 6 9 10 6 3
244)	1 3 6 9 10 7 3	245)	1 3 6 9 10 8 3	246)	1 3 6 9 10 9 3
247)	1 3 6 9 11 3 3	248)	1 3 6 9 11 4 3	249)	1 3 6 9 11 5 3
250)	1 3 6 9 11 6 3	251)	1 3 6 9 11 7 3	252)	1 3 6 9 11 8 3
253)	1 3 6 9 11 9 3	254)	1 3 6 9 12 3 3	255)	1 3 6 9 12 4 3
256)	1 3 6 9 12 5 3	257)	1 3 6 9 12 6 3	258)	1 3 6 9 12 7 3
259)	1 3 6 9 12 8 3	260)	1 3 6 9 12 9 3	261)	1 3 6 10 3 3 3
262)	1 3 6 10 4 3 3	263)	1 3 6 10 4 4 3	264)	1 3 6 10 5 3 3
265)	1 3 6 10 5 4 3	266)	1 3 6 10 5 5 3	267)	1 3 6 10 5 6 3
268)	1 3 6 10 6 3 3	269)	1 3 6 10 6 4 3	270)	1 3 6 10 6 5 3
271)	1 3 6 10 6 6 3	272)	1 3 6 10 6 7 3	273)	1 3 6 10 7 3 3
274)	1 3 6 10 7 4 3	275)	1 3 6 10 7 5 3	276)	1 3 6 10 7 6 3
277)	1 3 6 10 7 7 3	278)	1 3 6 10 7 8 3	279)	1 3 6 10 8 3 3
280)	1 3 6 10 8 4 3	281)	1 3 6 10 8 5 3	282)	1 3 6 10 8 6 3
283)	1 3 6 10 8 7 3	284)	1 3 6 10 8 8 3	285)	1 3 6 10 8 9 3
286)	1 3 6 10 9 3 3	287)	1 3 6 10 9 4 3	288)	1 3 6 10 9 5 3
289)	1 3 6 10 9 6 3	290)	1 3 6 10 9 7 3	291)	1 3 6 10 9 8 3
292)	1 3 6 10 9 9 3	293)	1 3 6 10 10 3 3	294)	1 3 6 10 10 4 3
295)	1 3 6 10 10 5 3	296)	1 3 6 10 10 6 3	297)	1 3 6 10 10 7 3
298)	1 3 6 10 10 8 3	299)	1 3 6 10 10 9 3	300)	1 3 6 10 11 3 3
301)	1 3 6 10 11 4 3	302)	1 3 6 10 11 5 3	303)	1 3 6 10 11 6 3
304)	1 3 6 10 11 7 3	305)	1 3 6 10 11 8 3	306)	1 3 6 10 11 9 3
307)	1 3 6 10 12 3 3	308)	1 3 6 10 12 4 3	309)	1 3 6 10 12 5 3

310)	1 3 6 10 12 6 3	311)	1 3 6 10 12 7 3	312)	1 3 6 10 12 8 3
313)	1 3 6 10 12 9 3	314)	1 3 6 10 13 3 3	315)	1 3 6 10 13 4 3
316)	1 3 6 10 13 5 3	317)	1 3 6 10 13 6 3	318)	1 3 6 10 13 7 3
319)	1 3 6 10 13 8 3	320)	1 3 6 10 13 9 3	321)	1 3 6 10 14 3 3
322)	1 3 6 10 14 4 3	323)	1 3 6 10 14 5 3	324)	1 3 6 10 14 6 3
325)	1 3 6 10 14 7 3	326)	1 3 6 10 14 8 3	327)	1 3 6 10 14 9 3
328)	1 3 6 10 15 3 3	329)	1 3 6 10 15 4 3	330)	1 3 6 10 15 5 3
331)	1 3 6 10 15 6 3	332)	1 3 6 10 15 7 3	333)	1 3 6 10 15 8 3
334)	1 3 6 10 15 9 3				

Table 7

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ABSTRACT

ARTINIAN O-SEQUENCE OF SOCLE DEGREE 6 AND TYPE 3

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Based on a theorem of Fröberg and Laksov in [7], 334 Codimension 3, Type 3, and Length 7 Artinian O-sequences was discovered by CoCoA (see Appendix A). Of them, after that, the case, that is not become a level, was studied .

After the O-sequence, is not become a level by Theorem and Remark, was found out, the table, which conforms to each case, was completed. Of the O-sequence that was not rejected by the former method, the special case, that is not become a level, was also demonstrated.

As a result, it was demonstrated that 221 O-sequences, of 334 O-sequences, is not a level. 10 O-sequences, that can not know whether they are a level or not by the former method, was remained.