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Artinian level algebra of codimension 3 and type 2

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이 논문을 석사학위논문으로 제출함.

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조지영의 석사학위 논문으로 인준함.

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목 차

논 문 개 요

Fröberg and Laksov의 정리에 근거하여 여차원이 3, 형태가 2, 길이가 7인 level이 될 가능성이 있는 341개의 Artinian O-수열에 대하여 level 여부를 연구하였다.

Theorem과 Remark를 통해 level이 되지 않는 O-수열을 찾아 각 경우에 해당하는 Table을 만들었다. 또한 앞의 방법을 통해 제거되지 않은 O-수열 중 level이 되지 않는 특별한 경우를 증명하였다.

level이 되는 58개의 수열은 CoCoA, S-plus를 이용하여 “link-sum”을 구성하였다.

1. **Introduction**

A Gorenstein algebra has been much studied, so we are interested in level Artinian O-sequences of type 2 with length 7.

Our main interest is to examine what kinds of Artinian O-sequences of type 2 can be level or not. We attempt to prove in several ways that codimension 3 Artinian O-sequences of type 2 and length 7 cannot be level.

It seems to be wise to start with the definition of Hilbert function. If we let $R = k[x_0, x_1, \ldots, x_n] = \bigoplus_{i\geq 0} R_i$, where k is an algebraically closed field of characteristic 0, and let I be a homogeneous ideal of R, $A = R/I$, then the ***Hilbert function*** of A, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, (or sometimes $\mathbf{H}(A, -)$) is defined by

$$\mathbf{H}_A(t) = \dim_k R_t - \dim_k I_t.$$

We consider standard Artinian algebras $A = R/I$, where I is a homogeneous ideal of R. The ***h-vector*** of A is $h(A) = (h_0, h_1, \ldots, h_\ell)$ where $h_i = \dim_k A_i = \dim_k R_i - \dim_k I_i$ and ℓ is the last index such that $\dim_k A_k \neq 0$. We call ℓ the ***socle degree*** A. Moreover, we shall assume that I does not contain any non-zero forms of degree 1 and n is defined as the ***codimension*** of A.

Using the algorithm in [9] from CoCoA [8] and a special case of a theorem of Fröberg and Laksov in [3], we have figured out that there are 341 Artinian O-sequences which could be level. To investigate these cases further, we consider one of those sequences, 1 3 5 4 4 4 2.

Let I be a lex-segment ideal of $R = k[x, y, z]$ with the Hilbert function

$1 \quad 3 \quad 5 \quad 4 \quad 4$. Then the minimal free resolution of R/I is

$$\begin{array}{lllllllll} 0 & \to & R^2(-5) \oplus R^4(-7) & \to & R^5(-4) \oplus R(-5) \oplus R^8(-6) & & & & \\ & \to & R(-2) \oplus R^3(-3) \oplus R(-4) \oplus R^4(-5) & \to & R & \to & R/I & \to & 0. \end{array}$$

This indicates that both copies $R(-5)$ of the last free module cannot be canceled. This means an Artinian algebra with Hilbert function having a sequence, $1 \quad 3 \quad 5 \quad 4 \quad 4$, cannot be level, and thus, by Theorem 8 in [13], any Artinian algebra with Hilbert function having the sequence, $1 \quad 3 \quad 5 \quad 4 \quad 4 \quad 4 \quad 2$, cannot be level, either.

We try to classify Artinian O-sequences of codimension 3 and type 2 with length 7 based on whether or not being level. In this thesis, we shall prove that some O-sequences among the above 341 cases, which look like possible level Artinian O-sequences, cannot be level Artinian O-sequences and produce some of them using 'Link-sum' constructions of sets of points in $\mathbb{P}^2$.

2. **Non-Level Artinian O-sequences of codimension 3 and type 2**

First of all, we introduce some definitions here.

Definition 2.1. (a) Let $h,\, i > 0$.
Then $h = \binom{m_i}{i} + \binom{m_i - 1}{i-1} + \cdots + \binom{m_j}{j}$ with $m_i > m_i - 1 > \cdots > m_j \geq 1$ is called ***i-th binomial expansion***.

(b) The sequnce $\{h_i\}_{i\geq 0}$ $(h_i \geq 0)$ is called an ***O-Sequence*** if $h_{i+1} \leq h_i^{\langle i \rangle},\ h_0 = 1,\ i \geq 1.$

Definition 2.2. Let $A = \bigoplus_{i\geq 0} A_i$ be graded ring. Then $A = k[A_1]$ is called ***G-algebra*** if A_1 is a finitely generated vector space over k.

Theorem 2.3 (Macaulay)**.** (a) $\{h_i\}_{i\geq 0}$ *is an O-Sequence.*

(b) $\{h_i\}_{i\geq 0}$ *is the Hilbert function for some standard G-algebra.*

Definition-Proposition 2.4 (Definition-Proposition 2.21 in [5])**.** Let $R = k[x_0, \ldots, x_n]$ and let $A = R/I$ be a Cohen-Macaulay ring of dimension d. Let

$$0 \to \mathcal{F}_{n-(d-1)} \to \cdots \to \mathcal{F}_1 \to I \to 0$$

be a minimal free resolution of I. Then

(a) If $B = B_0 \oplus \cdots \oplus B_\ell$ ($B_\ell \neq 0$) is an Artinian algebra, then B is ***level*** if and only if $B_\ell = \mathrm{Ann}(B_1)$.

(b) A is a ***level algebra*** if $\mathcal{F}_{n-(d-1)} = R^m(-s)$, for some $s > 0$. rank $\mathcal{F}_{n-(d-1)}$

= Cohen-Macaulay type of A.

(c) i) If $\mathbb{X}$ is a non-degenerate set of points in $\mathbb{P}^n$, $A = R/I_{\mathbb{X}}$ its coordinate ring, then we say that ℓ is the ***socle degree*** of $\mathbb{X}$ if ℓ is the socle degree of the Artinian algebra $B = A/\overline{L}A$, where $\overline{L}$ is any linear non-zero-divisor of A.

ii) $\mathbb{X}$ is called a ***level set*** of points if $A = R/I_{\mathbb{X}}$ is a level algebra. In this case, the socle degree of $\mathbb{X}$ is $\ell = \sigma(\mathbb{X}) + n - 1$.

(d) If $\overline{L}$ is a linear non-zero divisor in $A = R/I$, then A is level if and only if $A/\overline{L}A \simeq A/(L, I_{\mathbb{X}})$ is level.

(e) A 0-dimensional differentiable O-sequence (equivalently, an O-sequence whose first difference is the Hilbert function of an Artinian algebra) $b = \{b_i\}_{i \geq 0}$ with $b_1 = n + 1$, is called ***level*** if there is a level set of points in with Hilbert function b.

Definition 2.5. (a) A total order on the monomials of each degree is said to be a ***term order*** if

i) $x_1 > \cdots > x_n$, and

ii) $m_1 \geq m_2$ implies $mm_1 \geq mm_2$, for any monomials m, m_1 and m_2.

(b) The ***reverse lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $s < t \leq n$ and $i_s < j_s$.

(c) The ***lexicographic order*** is a term order defined to be $x_1^{i_1} \cdots x_n^{i_n} > x_1^{j_1} \cdots x_n^{j_n}$ if and only if

i) $\sum i_t > \sum j_t$ or

ii) $\sum i_t = \sum j_t$ and

there is s such that $i_t = j_t$ for $t < s \leq n$ and $i_s > j_s$.

(d) Let S be a subset of all monomials in R_d. S is a ***lex-segment*** if a monomial m of deree d is in S, then every monomial m' of degree d in R_d such that $m' > m$ is in S.

(e) Let $I = \bigoplus_{t \geq 0} I_t$ be a graded ideal of R. We say that I is a ***lex-segment ideal*** if for every $t \geq 0$, I_t is generated (as a vector space) by a lex-segment.

Theorem 2.6 (The Cancelation Principle, [1], [6]). *For any homogeneous ideal I and any i and d, there is a complex of $k \cong R/m-$modules $V_\bullet^d$ such that*

$$\begin{array}{rcl} V_i^d & \cong & \mathrm{Tor}_i^R(\mathrm{in}(I), k)_d \\ H_i(V_\bullet^d) & \cong & \mathrm{Tor}_i^R(I, k)_d. \end{array}$$

Remark 2.7. One way to paraphrase this theorem is to say that the minimal free resolution of I is obtained from that $\mathrm{in}(I)$, the ***initial ideal*** of I, by canceling some adjacent terms of the same degree.

The following 53 cases in Table 1 are not level by Theorem 2.6 and Remark 2.7.

44)	1 3 4 5 6 3 2	91)	1 3 5 6 6 3 2	96)	1 3 5 5 7 3 2
104)	1 3 5 7 4 3 2	112)	1 3 5 7 6 3 2	117)	1 3 5 7 7 3 2
122)	1 3 5 7 8 3 2	127)	1 3 5 7 9 3 2	128)	1 3 5 7 9 4 2
187)	1 3 6 7 4 3 2	195)	1 3 6 7 6 3 2	200)	1 3 6 7 7 3 2
205)	1 3 6 7 8 3 2	210)	1 3 6 7 9 3 2	211)	1 3 6 7 9 4 2
218)	1 3 6 8 4 3 2	222)	1 3 6 8 5 4 2	236)	1 3 6 8 8 3 2
217)	1 3 6 9 6 5 3	232)	1 3 6 9 8 9 3	242)	1 3 6 8 9 4 2
246)	1 3 6 8 10 3 2	247)	1 3 6 8 10 4 2	257)	1 3 6 9 5 3 2
258)	1 3 6 9 5 4 2	262)	1 3 6 9 6 3 2	264)	1 3 6 9 6 5 2
267)	1 3 6 9 7 3 2	272)	1 3 6 9 8 3 2	277)	1 3 6 9 9 3 2
278)	1 3 6 9 9 4 2	282)	1 3 6 9 10 3 2	283)	1 3 6 9 10 4 2
287)	1 3 6 9 11 3 2	288)	1 3 6 9 11 4 2	292)	1 3 6 9 12 3 2
294)	1 3 6 9 12 5 2	300)	1 3 6 10 4 3 2	303)	1 3 6 10 5 3 2
304)	1 3 6 10 5 4 2	308)	1 3 6 10 6 3 2	310)	1 3 6 10 6 5 2
313)	1 3 6 10 7 3 2	318)	1 3 6 10 8 3 2	323)	1 3 6 10 9 3 2
324)	1 3 6 10 9 4 2	328)	1 3 6 10 10 3 2	329)	1 3 6 10 10 4 2
333)	1 3 6 10 11 3 2	334)	1 3 6 10 11 4 2	338)	1 3 6 10 12 3 2
339)	1 3 6 10 12 4 2	340)	1 3 6 10 12 5 2		

Table 1

Theorem 2.8 ([2]). *Let* $R = k[x_1, x_2, x_3]$*and let* $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ *be the h-vector of a graded Artinian algebra* $A = R/I$ *with socle degree* s*. If*

$$h_{d-1} > h_d \qquad and \qquad h_d = h_{d+1} \leq 2d + 3$$

then **H** *is* **not** *level.*

Remark 2.9. Let **H** and R be as above. Then any Artinian algebra $A = R/I$ with Hilbert function **H** has a socle element in degree $d - 1$.

The following 148 cases in Table 2 are not level by Theorem 2.8.

1)	1 3 2 2 2 2 2	2)	1 3 3 2 2 2 2	3)	1 3 3 3 2 2 2
4)	1 3 3 3 3 2 2	6)	1 3 3 4 2 2 2	7)	1 3 3 4 3 2 2
8)	1 3 3 4 3 3 2	9)	1 3 3 4 4 2 2	12)	1 3 3 4 5 2 2
17)	1 3 4 2 2 2 2	18)	1 3 4 3 2 2 2	19)	1 3 4 3 3 2 2
20)	1 3 4 3 3 3 2	21)	1 3 4 4 2 2 2	22)	1 3 4 4 3 2 2
23)	1 3 4 4 3 2 2	24)	1 3 4 4 4 2 2	27)	1 3 4 4 5 2 2
32)	1 3 4 5 2 2 2	33)	1 3 4 5 3 2 2	34)	1 3 4 5 3 3 2
35)	1 3 4 5 4 2 2	37)	1 3 4 5 4 4 2	38)	1 3 4 5 5 2 2
43)	1 3 4 5 6 2 2	49)	1 3 5 3 2 2 2	50)	1 3 5 3 3 2 2
51)	1 3 5 3 3 3 2	52)	1 3 5 4 2 2 2	53)	1 3 5 4 3 2 2
54)	1 3 5 4 3 3 2	55)	1 3 5 4 4 2 2	56)	1 3 5 4 4 3 2
57)	1 3 5 4 4 4 2	58)	1 3 5 4 5 2 2	63)	1 3 5 5 2 2 2
64)	1 3 5 5 3 2 2	65)	1 3 5 5 3 3 2	66)	1 3 5 5 4 2 2
68)	1 3 5 5 4 4 2	69)	1 3 5 5 5 2 2	74)	1 3 5 5 6 2 2

TABLE 2

79)	1 3 5 6 2 2 2	80)	1 3 5 6 3 2 2	81)	1 3 5 6 3 3 2
82)	1 3 5 6 4 2 2	84)	1 3 5 6 4 4 2	85)	1 3 5 6 5 2 2
88)	1 3 5 6 5 5 2	90)	1 3 5 6 6 2 2	95)	1 3 5 6 7 2 2
100)	1 3 5 7 2 2 2	101)	1 3 5 7 3 2 2	102)	1 3 5 7 3 3 2
103)	1 3 5 7 4 2 2	105)	1 3 5 7 4 2 2	106)	1 3 5 7 5 2 2
109)	1 3 5 7 5 5 2	111)	1 3 5 7 6 2 2	115)	1 3 5 7 6 6 2
116)	1 3 5 7 7 2 2	121)	1 3 5 7 8 2 2	126)	1 3 5 7 9 2 2
131)	1 3 6 2 2 2 2	132)	1 3 6 3 2 2 2	133)	1 3 6 3 3 2 2
134)	1 3 6 3 3 3 2	135)	1 3 6 4 2 2 2	136)	1 3 6 4 3 2 2
137)	1 3 6 4 3 3 2	138)	1 3 6 4 4 2 2	139)	1 3 6 4 4 3 2
140)	1 3 6 4 4 4 2	141)	1 3 6 4 5 2 2	146)	1 3 6 5 2 2 2
147)	1 3 6 5 3 2 2	148)	1 3 6 5 3 3 2	149)	1 3 6 5 4 2 2
151)	1 3 6 5 4 4 2	152)	1 3 6 5 5 2 2	153)	1 3 6 5 5 3 2
154)	1 3 6 5 5 4 2	155)	1 3 6 5 5 5 2	156)	1 3 6 5 5 6 2
157)	1 3 6 5 6 2 2	162)	1 3 6 6 2 2 2	163)	1 3 6 6 3 2 2
164)	1 3 6 6 3 3 2	165)	1 3 6 6 4 2 2	167)	1 3 6 6 4 4 2
168)	1 3 6 6 5 2 2	171)	1 3 6 6 5 5 2	173)	1 3 6 6 6 2 2
178)	1 3 6 6 7 2 2	183)	1 3 6 7 2 2 2	184)	1 3 6 7 3 2 2
185)	1 3 6 7 3 3 2	186)	1 3 6 7 4 2 2	188)	1 3 6 7 4 4 2
189)	1 3 6 7 5 2 2	192)	1 3 6 7 5 5 2	194)	1 3 6 7 6 2 2
198)	1 3 6 7 6 6 2	199)	1 3 6 7 7 2 2	204)	1 3 6 7 8 2 2
209)	1 3 6 7 9 2 2	214)	1 3 6 8 2 2 2	215)	1 3 6 8 3 2 2
216)	1 3 6 8 3 3 2	217)	1 3 6 8 4 2 2	219)	1 3 6 8 4 4 2
220)	1 3 6 8 5 2 2	223)	1 3 6 8 5 5 2	225)	1 3 6 8 6 2 2
229)	1 3 6 8 6 6 2	230)	1 3 6 8 7 2 2	235)	1 3 6 8 8 2 2
240)	1 3 6 8 9 2 2	245)	1 3 6 8 10 2 2	250)	1 3 6 9 2 2 2
251)	1 3 6 9 3 2 2	252)	1 3 6 9 3 3 2	253)	1 3 6 9 4 2 2
255)	1 3 6 9 4 4 2	256)	1 3 6 9 5 2 2	259)	1 3 6 9 5 5 2
261)	1 3 6 9 6 2 2	265)	1 3 6 9 6 6 2	266)	1 3 6 9 7 2 2
271)	1 3 6 9 8 2 2	276)	1 3 6 9 9 2 2	286)	1 3 6 9 11 2 2
291)	1 3 6 9 12 2 2	296)	1 3 6 10 2 2 2	297)	1 3 6 10 3 2 2
298)	1 3 6 10 3 3 2	299)	1 3 6 10 4 2 2	301)	1 3 6 10 4 4 2
302)	1 3 6 10 5 2 2	305)	1 3 6 10 5 5 2	307)	1 3 6 10 6 2 2

Table 3

311)	1 3 6 10 6 6 2	312)	1 3 6 10 7 2 2	317)	1 3 6 10 8 2 2
322)	1 3 6 10 9 2 2	327)	1 3 6 10 10 2 2	332)	1 3 6 10 11 2 2
337)	1 3 6 10 12 2 2				

Table 4

Theorem 2.10 (Theorem 2.17, [4]). *Let $h_{d-2},\ h_{d-1},\ h_d$ be three non-zero integers such that*

$$h_d = h_{d-1}^{\langle d-1\rangle} \qquad \text{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2\rangle}\ .$$

Let I be any ideal in $R = k[x_1, \ldots, x_n]$ such that the Hilbert function of R/I satisfies

$$\begin{array}{lcl} \mathbf{H}(R/I, d-2) &=& h_{d-2} + \varepsilon, \quad \varepsilon \geq 0 \\ \mathbf{H}(R/I, d-1) &=& h_{d-1}, \\ \mathbf{H}(R/I, d) &=& h_d\,. \end{array}$$

Then, the ring R/I has socle of dimension ε in degree $d-2$.

Using Theorem 2.10, one can show that the following 43 cases in Table 5 are not level O-sequences.

10)	1 3 3 4 4 3 2	11)	1 3 3 4 4 4 2	13)	1 3 3 4 5 3 2
14)	1 3 3 4 5 4 2	15)	1 3 4 4 5 5 2	16)	1 3 4 4 5 6 2
28)	1 3 4 4 5 3 2	29)	1 3 4 4 5 4 2	30)	1 3 4 4 5 5 2
31)	1 3 4 4 5 6 2	42)	1 3 4 5 5 6 2	59)	1 3 5 4 5 3 2
60)	1 3 5 4 5 4 2	61)	1 3 5 4 5 5 2	62)	1 3 5 4 5 6 2
73)	1 3 5 5 5 6 2	75)	1 3 5 5 6 3 2	76)	1 3 5 5 6 4 2
77)	1 3 5 5 6 5 2	78)	1 3 5 5 6 6 2	89)	1 3 5 6 5 6 2
110)	1 3 5 7 5 6 2	142)	1 3 6 4 5 3 2	143)	1 3 6 4 5 4 2
144)	1 3 6 4 5 5 2	145)	1 3 6 4 5 6 2	150)	1 3 6 5 4 3 2
158)	1 3 6 5 6 3 2	159)	1 3 6 5 6 4 2	160)	1 3 6 5 6 5 2

Table 5

161)	1 3 6 5 6 6 2	172)	1 3 6 6 5 6 2	179)	1 3 6 6 7 3 2
180)	1 3 6 6 7 4 2	181)	1 3 6 6 7 5 2	182)	1 3 6 6 7 6 2
193)	1 3 6 7 5 6 2	212)	1 3 6 7 9 5 2	213)	1 3 6 7 9 6 2
224)	1 3 6 8 5 6 2	260)	1 3 6 9 5 6 2	306)	1 3 6 10 5 6 2
311)	1 3 6 10 6 6 2				

Table 6

Example 2.11. There are no Artinian level algebra of socle degree s with Hilbert function (1, 3, . . . , 5, 3, 2).

Hence the following cases in Table 7 are not level.

39)	1 3 4 5 5 3 2	70)	1 3 5 5 5 3 2	86)	1 3 5 6 5 3 2
107)	1 3 5 7 5 3 2	190)	1 3 6 7 5 3 2	221)	1 3 6 8 5 3 2

Table 7

3. **Inverse System**

We now recall an interesting method for construsting Artinian level algebras. This method is based on the idea of ***Macaulay's Inverse Systems***. We will only give a quick review of the method, and refer the reader to in the book [4] for more details.

Let $R = k[x_1, \cdots, x_n]$ and $S = k[y_1, \cdots, y_n]$. We can consider S as a graded R-module by: if $F \in S_j$ then $x_i \circ F = \frac{\partial}{\partial y_i} F$. We extend this action in the obvious way, and note that the action *lowers* the degree of a given form on S and hence S is not a finitely generated R-module.

There is an inclusion reversing function from the ideals of R to the R-submodules of S defined by:

$$\varphi_1 : \{\text{ideals of } R\} \to \{R\text{-submodules of } S\}$$

where

$$\varphi_1(I) = \{F \in S | G \circ F = 0 \text{ for all } G \in I\}$$

This is a $1-1$ correspondence whose inverse φ_2 is given by $\varphi_2(M) = \text{ann}_R(M) = \{r \in R \mid r \cdot x = 0, \forall x \in M\}$. In fact, we denote $\varphi_1(I)$ by I^{-1}, which is called the ***inverse system*** to I.

It is very easy to construct I^{-1} (and this is at the heart of the proof of the 1-1 correspondence). First observe that the pairing

$$R_j \times S_j \longrightarrow S_0 \simeq k$$

is a perfect pairing, and so S_j can be identified with R_j^* (the dual vector space to R_j). If V is a subspace of R_j we write $V^\perp$ for the annihilator of V in this pairing. If $I \subset R$ is an ideal and I_j its j^{th} graded piece, then Macaulay observed that:

$$(I^{-1})_j = I_J^\perp.$$

It follows immediately that

$$\dim_k (I^{-1})_j = \dim_k R_j - \dim_k I_j = \mathbf{H}(R/I, j).$$

It is a simple consequence of this last observation that I^{-1} is a finitely generated R-submodule of S if and only if R/I is Artinian.

Remark 3.1. There is another way to define Inverse Systems which considers S as an R-module in a different way. In this other method, we consider the ***contraction*** operations, D_{x_i} where, if F is a monomial in S_j then

$$D_{x_i}(F) = \begin{cases} 0, & \text{if } y_i \text{ does not divide } F, \\ F/y_i & \text{if } y_i \text{ divides } F. \end{cases}$$

We extend this action to all of S in the obvious way and recall that when the characteristic of k is 0, this action is equivalent to the one described above. The contraction operation has the advantage that it doesn't end up increasing the sizes of coefficients.

The really interesting connection between inverse systems and what we've been considering is the following theorem of Macaulay. We continue with notations as above.

Theorem 3.2 (Theorem 5.2, [5])**.** *Let I be an Artinian ideal of R and I^{-1} its inverse system. Then I^{-1} has exactly ν_j minimal generators of degree j if and only if the socle of R/I in degree j has dimension exactly ν_j.*

Lemma 3.3. *Let $R = k[x, y, z]$. If I is a homogeneous ideal of R such that R/I has the Hilbert function $\mathbf{H} = (1, 3, 4, 5, \dots)$, then two minimal generators of I in degree 2 have a linear common factor.*

Proof. Let I be an ideal in $R = k[x, y, z]$ so that the Hilbert function of $A = R/I$ begins

$$1 \quad 3 \quad 4 \quad 5 \quad \cdots$$

Let $\langle F, G\rangle = I_2$ and assume that F, G is a regular sequence. Since 4 and 5 have a maximal growth in degrees 2 and 3, we see that I does not have any generators in degree 3, that is,

$$\begin{aligned} R_1 I_2 &= I_3 \\ &= \langle Fx, Fy, Fz, Gx, Gy, Gz\rangle. \end{aligned}$$

Since $\dim_k I_3 = 5 = 10 - \mathbf{H}(R/I, 3)$, we have that one of Fx, Fy, Fz, Gx, Gy and Gz is a linear combination of the rest of 5 elements.

Without loss of generality, we may assume that

$$Fx \in \langle Fy, Fz, Gx, Gy, Gz\rangle.$$

Then

$$Fx = aFy + bFz + cGx + dGy + eGz$$

where a, b, c, d, $e \in k$, and so

$$F \cdot (x - ay - bz) = G \cdot (cx + dy + ez) \in (G)$$

Since F, G is a regular sequence, we have that

$$(x - ay - bz) \in (G),$$

which is a contradiction since $\deg(G) = 2$. In other words, F and G cannot be a regular sequence.

Note that $k[x, y, z]$ is a unique factorization domain. Hence

$$(x - ay - bz) \mid F \cdot (x - ay - bz)$$

implies

$$(x - ay - bz) \mid G \quad \text{or} \quad (x - ay - bz) \mid (cx + dy + ez).$$

However, if $(x - ay - bz) \mid (cx + dy + ez)$, then $F = \alpha \cdot G$ for some $\alpha \in k - (0)$, that is, $\dim_k \langle F, G \rangle = 1$, a contradiction.

Thus $(x - ay - bz) \mid G$, that is, $G = (x - ay - bz) \cdot L$ for some linear form $L \in R_1$ and so

$$F \cdot (x - ay - bz) = G \cdot (cx + dy + ez)$$

$$= (cx + dy + ez) \cdot L \cdot (x - ay - bz)$$

This implies that $F = (cx + dy + ez) \cdot L$.

In other words, F and G have a linear common factor $L \in R_1$, as we wished. □

Proposition 3.4. *There is no Artinian level algebra with Hilbert function, H= (1, 3, 4, 5, 6, 6, 2).*

Proof. Suppose there is an Artinian level algebra $A = R/I$ with Hilbert function H. Then $I^{-1} = \langle F, G \rangle$ where $F, G \in S_6$.

By Lemma 3.3, we may assume that $I_2 = \langle xy, xz \rangle$ or $\langle x^2, xy \rangle$.

Case 1. $I_2 = \langle xy, xz \rangle$.

If $F = \cdots + XYH(X,Y,Z) + \cdots$ where H is a monomial in S_4. Then $H(x,y,z)xy \in I_6$. $H(x,y,z)xy \circ F = 0 + \cdots + xyH(x,y,z) \circ XYH(X,Y,Z) + 0 + \cdots = xyH(x,y,z) \circ XYH(X,Y,Z) \neq 0$. In other words, any monomial divided by XY cannot be a term of F. Similarly, any term divided by XZ cannot be a term of F.

So if $I^{-1} = \langle F, G \rangle$ and $F, G \in S_6$, then

$$\begin{aligned} F &= aX^6 + F_1 \\ G &= bX^6 + G_1 \end{aligned}$$

where $F_1, G_1 \in k[Y, Z]$ and $a, b \in F$.

If $a = b = 0$, then $F, G \in k[Y, Z]$, that is, $I = \langle F, G \rangle^{-1}$ contains x. Hence $H(R/I, 1) \leq 2$, a contradiction. Thus we may assume that either a or b is not 0. If $a \neq 0$, then $G' = G - a^{-1}bF \in k[Y, Z]$, and $\langle F, G \rangle = \langle F, G' \rangle$. Thus, without loss of generality, we may assume $F = X^6 + F_1$, where $F_1 \in k[Y, Z]$ and $G \in k[Y, Z]$.

Note that G is a polynomial in only two variables, and so can have at most 2 linearly independent first derivatives. Since F can also have at most 3 linearly independent first derivatives, we see that F and G can generate at most a subspace of S_5 of dimension at most 5. But, $H(R/I, 5) = 6$ and so this is impossible.

Case 2. $I_2 = \langle x^2, xy \rangle$.

By the similar argument as in Case 1, any monomial divided by X^2 or XY in S_6 cannot be a term of F or G. Hence

$$\begin{aligned} F &= aXZ^5 + F_1, \quad \text{where} \quad F_1 \in k[Y, Z], a \in k \\ G &= bXZ^5 + G_1, \quad \text{where} \quad G_1 \in k[Y, Z], b \in k \end{aligned}$$

By the same idea as in Case 1 again, we may assume that $a = 1$, $b = 0$. But, then G is a polynomial in only two variables and so can have at most 2 linearly independent first derivatives. Since F can have at most 3 linearly independent first derivatives, we see that F and G

can generate at most a subspace of S_5 of dimension at most 5. But, $H(R/I,5)=6$ and so this is impossible. Therefore, by Cases 1 and 2, an Artinian level algebra with Hilbert function dose not exist, as we wished. □

Using the inverse system, as in proposition 3.4, one can show that the following cases cannot be level.

45)	1 3 4 5 6 4 2	47)	1 3 4 5 6 6 2	67)	1 3 5 5 4 3 2
83)	1 3 5 6 4 3 2	97)	1 3 5 6 7 4 2	118)	1 3 5 7 7 4 2
170)	1 3 6 6 5 4 2	191)	1 3 6 7 5 4 2		

TABLE 8

Corollary 3.5 (Corollary 2.11, [5]). *Let $H=(1,h_1,\cdots,h_s)$ be a level sequence with $h_s\geq 1$ and let A be any level Artinian algebra with $h(A)=h$. Then the vector $(h_s,h_s-1,\cdots,1)$ is the sum of*

1) a Gorenstein sequence $(1,b_1,\cdots,b_s-1,1)$ $of lengths+1)$ which is the $h-vector$ of a quotient of A and

2) h_s-1 O-sequences $1,d_j1,\cdots,d_jrj)$, $j=1,\cdots,h_s-1$, each of which is the $h-vector$ of a quotient of A.

Corollary 3.6. *There is no Artinian level algebra with Hilbert function, in the following cases.*

$(1,3,\ldots,\geq 8,4,2)$, $(1,3,\ldots,\geq 10,5,2)$, $(1,3,6,\ldots,4,3,2)$, $(1,3,6,\ldots,5,3,2)$, $(1,3,\ldots,2,2)$

Proof. . We shall prove that $(1, 3, \dots, 8, 4, 2)$ cannot be level.

Assume $(1, 3, \dots, 8, 4, 2)$ is level.

Case 1. $(2, 4, 8, \dots, 3, 1) = (1, 1, 1, \dots, 1, 1) + (1, 3, 7, \dots, 2, 0)$, is impossible since $1, 3, 7, \dots$ is not an O-sequence.

Case 2. $(2, 4, 8, \dots, 3, 1) = (1, 2, \alpha, \dots, \alpha, 2, 1) + (1, 2, 8-\alpha, \dots)$ where $\alpha = 2, 3$. But then, $8-\alpha = 7$ or 6 is not possible, since $(1, 2, \geq 6, \dots)$is not an O-sequence.

Case 3. $(2, 4, 8, \dots, 3, 1) = (1, 3, \alpha, \dots, \alpha, 3, 1) + (1, 1, 8-\alpha, \dots)$ where $\alpha = 3, 4, 5, 6$. But then, $8 - \alpha = 2, 3, 4, 5$ is not possible, since $(1, 1, \geq 3, \dots)$ is not an O-sequence.

Therefore, $(1, 3, \dots, 8, 4, 2)$ cannot be level. Using Corollary 3.6, one can show that the following cases in Table 9 are not level, as we desired. □

123)	1 3 5 7 8 4 2	166)	1 3 6 6 4 3 2	206)	1 3 6 7 8 4 2
237)	1 3 6 8 8 4 2	248)	1 3 6 8 10 5 2	254)	1 3 6 9 4 3 2
273)	1 3 6 9 8 4 2	281)	1 3 6 9 10 2 2	284)	1 3 6 9 10 5 2
289)	1 3 6 9 11 5 2	293)	1 3 6 9 12 4 2	319)	1 3 6 10 8 4 2
330)	1 3 6 10 10 5 2	335)	1 3 6 10 11 5 2		

TABLE 9

Proposition 3.7 (Proposition 3.9, [5]). *Let h_{d-2}, h_{d-1}, h_d be three integers such that*

$$h_d = h_{d-1}^{\langle d-1 \rangle} \qquad \textit{and} \qquad h_{d-1} = h_{d-2}^{\langle d-2 \rangle} .$$

Let I be any ideal in $R = k[x, y, z]$ for which

$$\begin{array}{lcl} \mathbf{H}(R/I, d-2) & = & h_{d-2} + \varepsilon, \quad \varepsilon \geq 3, \\ \mathbf{H}(R/I, d-1) & = & h_{d-1}, \\ \mathbf{H}(R/I, d) & = & h_d - 1\,. \end{array}$$

Then $\dim_k soc(R/I)_{d-2} \geq 1$ *and so any O-sequence*

$$(1, n, \ldots, h_{d-2} + \varepsilon, h_{d-1}, h_d - 1, \ldots, h_s)$$

of length $s + 1$ $(s \geq d)$ is not a level O-sequence.

Corollary 3.8. *There is no Artinian level algebra with Hilbert function* $(1, 3, \alpha, \beta, \geq 6, 3, 2)$

Proof. Note that 3,3,3 has a maximal growth in degrees 4, 5, and 6. So

$$6_4 = 6 = 3 + 3 \text{ and}$$
$$2_6 = 2 = 3 - 1$$

and hence, by Proposition 3.7, any O-sequence $(1, 3, \alpha, \beta, \geq 6, 3, 2)$ cannot be level, as we wished.

Using Corollary3.8, the following cases in Table 10 are not level. □

226)	1 3 6 8 6 3 2	231)	1 3 6 8 7 3 2	241)	1 3 6 8 9 3 2

TABLE 10

Proposition 3.9. *There is no Artinian level algebra with Hilbert function* $(1, 3, 5, 7, 6, 6, 2)$.

Proof. The Betti diagram of the lex-segment ideal with Hilbert function $(1, 3, 5, 7, 6, 6, 2)$ is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & - & - & - \\ - & \beta_{0,2} & - & - \\ - & - & - & - \\ - & \beta_{0,4} & \beta_{1,5} & \beta_{2,6} \\ - & \beta_{0,5} & \beta_{1,6} & \beta_{2,7} \\ - & \beta_{0,6} & \beta_{1,7} & \beta_{2,8} \\ - & \beta_{0,7} & \beta_{1,8} & \beta_{2,9} \end{pmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & - & - \\ - & 3 & 5 & 2 \\ - & 1 & 2 & 1 \\ - & 5 & 9 & 4 \\ - & 2 & 4 & 2 \end{pmatrix} \end{array}$$

Now suppose that $A = R/I,\ R = k[x_1, x_2, x_3]$ is a level algebra with Hilbert function as above.

Claim. The minimal number of generators of I, in degree 6, is < 5.

Notice that once this claim is proved we are done. Since then $\beta_{0,6}(A) < 5$ and by the Bigatti-Hulett-Pardue result we must have $\beta_{1,6}(A) < 2$. This, in turn, implies that $\beta_{2,6}(A) \geq 1$ and hence A cannot be level.

Proof of Claim. Suppose that I has 5 generators in degree 6 and let $J = I_{\leq 5}$. Then the Hilbert function of R/J begins $1\ 3\ 5\ 7\ 6\ 6\ 7\ \cdots$. Then 5, 6, 7 in degrees 4, 5 and 6 has a maximal growth, and hence, by Theorem 2.10, R/J has a socle element in degree 4. Since R/J and R/I agree in degree ≤ 5 we conclude that R/I cannot be level. $\square$

We note that, using the same kind of argument, one can show that the following Hilbert function are also not level sequences:

228)	1 3 6 8 6 5 2	270)	1 3 6 9 7 6 2	309)	1 3 6 10 6 4 2
316)	1 3 6 10 7 6 2				

Table 11

Proposition 3.10. *There is no Artinian level algebra with Hilbert function* $(1, 3, 6, 9, 6, 4, 2)$.

Proof. The Betti diagram for the lex-segment ideal with this Hilbert function $(1, 3, 6, 9, 6, 4, 2)$ is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \quad\ 0 \quad 1 \quad 2 \\ \begin{pmatrix} 1 & - & - & - \\ - & - & - & - \\ - & 1 & - & - \\ - & 6 & 10 & 4 \\ - & 3 & 5 & 2 \\ - & 2 & 4 & 2 \\ - & 2 & 4 & 2 \end{pmatrix}\end{array}$$

If there were a level algebra with this h-vector we'd have to be able to cancel $\beta_{2,6} = 4$ from this diagram.

So, suppose that $A = R/I$ is a level algebra with this h-vector. If I has no generators in degree 6 then we'd have a contradiction: for then

$\beta_{0,6} = 0$ and so $\beta_{1,6}(A) \leq 3$. But, we need $\beta_{1,6}(A) \geq 4$ to cancel the $\beta_{2,6} = 4$.

So, it suffices to show that I has no generators in degree 6 (we know that it has ≤ 2 generators in degree 6).

Case 1. Suppose I has two generators in degree 6. Let $J = \langle I_{\leq 5} \rangle$. Then the Hilbert function of R/J begins 1 3 6 9 6 4 4 $\cdots$. By Theorem2.10, R/J has a 2-dimensional socle in degree 4 and hence so does I. That is a contradiction.

Case 2. Now suppose that I has one generator in degree 6 and let $J = \langle I_{\leq 5} \rangle$ as above. Then the Hilbert function of $B = R/J$ begins 1 3 6 9 6 4 3 $t \cdots$. We don't know exactly what t is but we can say that $0 \leq t \leq 3$.

Let's first consider the possibility that $t = 3$. Since $J \subseteq I$ we have a canonical surjection R/J onto R/I, which is an isomorphism in degree ≤ 5. By Theorem2.10, R/J has non-zero socle in degree 5. The surjection (which is an isomorphism in degree 5) carries this non-zero socle into non-zero socle of R/I in degree 5, which is a contradiction. So, assume that $t \leq 2$. The lex-segment ideal whose h-vector is

$h = (1, 3, 6, 9, 6, 4, 3, t)$ has Betti diagram which starts

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{pmatrix} & 0 & 1 & 2 \\ 1 & - & - & - \\ - & - & - & - \\ - & 1 & - & - \\ - & 6 & 10 & 4 \\ - & 3 & 5 & 2 \\ - & 1 & 2 & 1 \\ - & 3-t & 6-2t & 3-t \\ & & & \end{pmatrix}$$

Since A is level, $\beta_{2,7}(A) = 0$ and since A and B agree in degree less than or equal to 5, that implies that B has no socle in degree 4 either, so $\beta_{2,7}(B) = 0$ as well. That implies that $\beta_{1,7}(B) = 0$ as well. This, in turn, implies that $\beta_{0,7}(B) = 3 - t$ exactly. But $3 - t > 0$ and so J has generators in degree 7, which is a contradiction. □

In an entirely similar way we can show that the following Hilbert function are also not level sequences:

129)	1 3 5 7 9 5 2	263)	1 3 6 9 6 4 2	315)	1 3 6 10 7 5 2

TABLE 12

4. **The Construction of Some Level O-sequences of Type 2 and Socle degree 7.**

In this appendix, we introduce how to construct level O-sequences using the “link-sum” construction which are given by Cho in [9] and Yoo in [12]. They used the computer programs S-plus and CoCoA to produce them here.

Definition 4.1 (Definition 2.1, [6])**.** (a) A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a ***basic configuration*** of type (d, e) if there exists distinct elements b_j, c_j in k such that

$$I_{\mathbb{X}} = \left(\prod_{j=1}^{d}(x - b_j z), \prod_{j=1}^{e}(y - c_j z) \right).$$

We denote $\mathbb{X} := \mathbb{B}(d, e)$.

(b) A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a ***pure configuration*** if there exist finite basic configurations $\mathbb{B}(d_1, e_1), \ldots, \mathbb{B}(d_m, e_m)$ where $e_1 > \cdots > e_m$, which satisfy the following three conditions:

i) $\mathbb{B}(d_i, e_i) \cap \mathbb{B}(d_j, e_j) = \phi$ if $i \neq j$,

ii) $\mathbb{X} = \mathbb{B}(d_1, e_1) \cup \cdots \cup \mathbb{B}(d_m, e_m)$,

iii) $\varphi(\mathbb{B}(d_i, e_i)) \supset \varphi(\mathbb{B}(d_{i+1}, e_{i+1}))$ for all $1 \leq i \leq m-1$, where $\varphi : \mathbb{P}^2 \setminus \{(1, 0, 0)\} \to \mathbb{P}^1$ is the map defined by sending the point (x, y, z) to the point (y, z). In this case, we denote $\mathbb{X} = \cup_{i=1}^{m} \mathbb{B}(d_i, e_i)$.

Proposition 4.2 (Proposition 3.8, [10]). *Let* $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ *be a pure configuration in* $\mathbb{P}^2$*. Then a minimal free resolution of* $\mathbb{X}$ *is :*

$$0 \to \bigoplus_{i=1}^{m} R(-p_i) \to \bigoplus_{i=1}^{m+1} R(-q_i) \to R \to R/I_{\mathbb{X}} \to 0,$$

where

$$q_1 = e_1, \quad q_i = d_1 + \cdots + d_{i-1} + e_i \quad (2 \leqslant i \leqslant m),$$

$$q_{m+1} = d_1 + \cdots + d_m, \quad p_i = q_i + d_i \quad (1 \leqslant i \leqslant m).$$

Example 4.3 (Example 3.9, [10]). Let $\mathbb{X} = \mathbb{B}(3,6) \cup \mathbb{B}(4,2) \cup \mathbb{B}(2,2) \cup \mathbb{B}(1,1)$ be a pure configuration in $\mathbb{P}^2$.

$$\mathbb{Z} = \left\{ \begin{array}{cccccccc} \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \right\}$$

Then, by proposition 4.2, the minimal free resolution of $\mathbb{Z}$ is

$0 \to R^4(-9) \to R(-6) \oplus R^2(-7) \oplus R^2(-8) \to R \to R/I \to 0.$

Corollary 4.4 (Corollary 3.10, [5]). *Let* $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ *be a pure configuration in* $\mathbb{P}^2$*. Then* $\mathbb{X}$ *is level if and only if*

$$e_i - e_{i+1} = d_{i+1}$$

for all $1 \leqslant i \leqslant m-1$*.*

If $r(A)$ is the Cohen-Macaulay type of a Cohen-Macaulay standard graded k-algebra A, and if $A = \bigoplus_{i\geqslant 0} A_i$ is an Artinian level algebra, then $r(A) = dim_k A_{\sigma(A)-1}$, where $\sigma(A) = min\{i | A_i = 0\}$. If $\mathbb{Z} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ is a level pure configuration in $\mathbb{P}^2$, then $r(\mathbb{Z}) = r(R/I_{\mathbb{Z}}) = m$.

Lemma 4.5 (Lemma 3.14, [5])**.** *Let $\mathbb{Z}$ be a level set of points in $\mathbb{P}^n$ and $\mathbb{X}$ a subset of $\mathbb{Z}$. Set $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is an Artinion level graded k-algebra with $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = \sigma(\mathbb{Z}) - 1$ and $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leqslant r(\mathbb{Z})$.*

Corollary 4.6 (Corollary 3.15, [5])**.** *Let $\mathbb{Z} = \bigcup_{i=1}^{m} \mathbb{B}(d_i, e_i)$ be a level pure configuration in $\mathbb{P}^2$ and $\mathbb{X}$ a subset of $\mathbb{Z}$. Set $\mathbb{Y} := \mathbb{Z}/\mathbb{X}$. Then $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is an Artinion level graded k-algebra with $\sigma(R/(I_{\mathbb{X}}+I_{\mathbb{Y}})) = d_1 + e_1 - 2$ and $r(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) \leqslant m$.*

Let $\mathbb{X}$ be the set of all $\bullet$'s in $\mathbb{Z}$ and $\mathbb{Y}$ be the set of all $*$'s in $\mathbb{Z}$. Next to each diagram we give the Hilbert functions of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $A = R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$.

5) [1,3,3,3,3,3,2]

$$\mathbb{Z} = \left\{\begin{matrix} \bullet & * & * & & & \\ * & * & * & & & \\ \bullet & * & * & & & \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & \bullet & * & * & * \end{matrix}\right\} \quad \begin{matrix} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 21 & 24 & 24 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 3 & 3 & 3 & 3 & 2 & 0 & \rightarrow . \end{matrix}$$

25) [1,3,4,4,4,3,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \bullet & \ast & \ast \\ \ast & \bullet & \ast & \bullet & \ast & \ast \\ \ast & \ast & \ast & \bullet & \ast & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 23 & 23 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 4 & 4 & 4 & 3 & 2 & 0 & \rightarrow . \end{array}$$

26) [1,3,4,4,4,4,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \bullet & \ast & \ast & \ast & \bullet \\ \ast & \ast & \bullet & \ast & \ast & \bullet \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 21 & 23 & 23 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 4 & 4 & 4 & 4 & 2 & 0 & \rightarrow . \end{array}$$

36) [1,3,4,5,4,3,2]

$$\mathbb{Z} = \left\{\begin{array}{ccccc} \ast & \ast & \ast & & \\ \ast & \ast & \ast & & \\ \ast & \ast & \ast & \bullet & \ast \\ \bullet & \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 21 & 21 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 4 & 5 & 4 & 3 & 2 & 0 & \rightarrow . \end{array}$$

40) [1,3,4,5,6,5,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \bullet & & & \\ \ast & \ast & \ast & \bullet & \ast & \ast \\ \ast & \ast & \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \ast & \bullet \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 22 & 22 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 4 & 5 & 6 & 5 & 2 & 0 & \rightarrow . \end{array}$$

41) [1,3,4,5,5,5,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \ast & \bullet & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \bullet & \ast & \bullet & \bullet & \bullet & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 21 & 22 & 22 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 4 & 5 & 5 & 5 & 2 & 0 & \rightarrow . \end{array}$$

46) [1,3,4,5,6,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ast \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 21 & 21 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 4 & 5 & 6 & 5 & 2 & 0 & \rightarrow & . \end{array}$$

71) [1,3,5,5,5,4,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \bullet & \bullet & \bullet & & & \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \bullet & \ast & \ast & \bullet & \ast \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 22 & 22 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 5 & 5 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

72) [1,3,5,5,5,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \bullet & & & \\ \ast & \ast & \ast & & & \\ \bullet & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \bullet & \bullet & \ast & \bullet & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 21 & 22 & 22 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 5 & 5 & 5 & 2 & 0 & \rightarrow & . \end{array}$$

87) [1,3,5,6,5,4,2]

$$\mathbb{Z}=\left\{\begin{array}{ccccc} \ast & \ast & \ast & & \\ \ast & \ast & \ast & & \\ \ast & \ast & \ast & \ast & \bullet \\ \ast & \ast & \bullet & \ast & \bullet \\ \ast & \ast & \bullet & \ast & \bullet \\ \ast & \ast & \ast & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 20 & 20 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 6 & 5 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

92) [1,3,5,6,6,4,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \bullet & \ast & \ast & & & \\ \ast & \bullet & \ast & & & \\ \bullet & \bullet & \bullet & & & \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 21 & 21 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 6 & 6 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

93) [1,3,5,6,6,5,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \bullet & \ast & \ast & \ast & \ast & \ast \\ \bullet & \ast & \ast & \bullet & \bullet & \bullet \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 21 & 21 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 6 & 6 & 5 & 2 & 0 & \rightarrow . \end{array}$$

94) [1,3,5,6,6,6,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \bullet & \ast & \bullet & & & \\ \ast & \ast & \bullet & & & \\ \ast & \ast & \bullet & \ast & \ast & \ast \\ \ast & \ast & \bullet & \ast & \ast & \ast \\ \ast & \ast & \ast & \bullet & \ast & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 21 & 21 & 21 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 6 & 6 & 6 & 2 & 0 & \rightarrow . \end{array}$$

98) [1,3,5,6,7,5,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \bullet & \ast & & & \\ \bullet & \ast & \bullet & & & \\ \ast & \bullet & \ast & & & \\ \ast & \ast & \bullet & \ast & \ast & \ast \\ \ast & \ast & \ast & \bullet & \ast & \ast \\ \ast & \ast & \ast & \ast & \bullet & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 20 & 20 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 6 & 7 & 5 & 2 & 0 & \rightarrow . \end{array}$$

99) [1,3,5,6,7,6,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \bullet & \ast & & & \\ \bullet & \ast & \ast & & & \\ \ast & \bullet & \ast & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \ast & \bullet & \ast & \ast & \bullet & \ast \\ \ast & \bullet & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 6 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 20 & 20 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 6 & 7 & 6 & 2 & 0 & \rightarrow . \end{array}$$

113) [1,3,5,7,6,4,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \ast & \bullet & \bullet & & & \\ \ast & \bullet & \bullet & & & \\ \ast & \bullet & \bullet & & & \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 7 & 7 & 7 & 7 & 7 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 20 & 20 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 7 & 6 & 4 & 2 & 0 & \rightarrow . \end{array}$$

114) [1,3,5,7,6,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\bullet&\bullet&\bullet&\bullet&&\\ \ast&\ast&\ast&\bullet&&\\ \ast&\ast&\ast&\bullet&\ast&\ast\\ \ast&\ast&\ast&\bullet&\ast&\ast\\ \ast&\ast&\ast&\ast&\ast&\ast\end{array}\right\}\begin{array}{rrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 24 & 26 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 7 & 7 & 7 & 7 & 7 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 19 & 19 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 5 & 7 & 6 & 5 & 2 & 0 & \rightarrow .\end{array}$$

119) [1,3,5,7,7,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\ast&\bullet&\ast&&&\\ \ast&\ast&\ast&&&\\ \ast&\ast&\ast&&&\\ \ast&\bullet&\ast&\ast&\bullet&\ast\\ \ast&\bullet&\ast&\ast&\bullet&\ast\\ \ast&\bullet&\ast&\ast&\bullet&\ast\end{array}\right\}\begin{array}{rrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 7 & 7 & 7 & 7 & 7 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 20 & 20 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 5 & 7 & 7 & 5 & 2 & 0 & \rightarrow .\end{array}$$

120) [1,3,5,7,7,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\ast&\ast&\ast&&&\\ \ast&\ast&\bullet&&&\\ \ast&\ast&\bullet&&&\\ \bullet&\bullet&\ast&\ast&\bullet&\bullet\\ \ast&\ast&\ast&\ast&\ast&\ast\\ \ast&\ast&\bullet&\ast&\ast&\ast\end{array}\right\}\begin{array}{rrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 7 & 7 & 7 & 7 & 7 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 20 & 20 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 5 & 7 & 7 & 6 & 2 & 0 & \rightarrow .\end{array}$$

124) [1,3,5,7,8,5,2]

$$\mathbb{Z}=\left\{\begin{array}{ccccc}\ast&\bullet&&&\\ \ast&\ast&&&\\ \ast&\bullet&&&\\ \ast&\ast&\ast&\ast&\ast\\ \ast&\bullet&\ast&\ast&\ast\\ \ast&\bullet&\bullet&\bullet&\bullet\\ \ast&\bullet&\ast&\ast&\ast\end{array}\right\}\begin{array}{rrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 7 & 8 & 8 & 8 & 8 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 18 & 18 & 18 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 5 & 7 & 8 & 5 & 2 & 0 & \rightarrow .\end{array}$$

125) [1,3,5,7,8,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\ast&\ast&\bullet&&&\\ \ast&\bullet&\ast&&&\\ \ast&\ast&\bullet&&&\\ \ast&\ast&\ast&\bullet&\ast&\ast\\ \ast&\ast&\bullet&\ast&\bullet&\ast\\ \ast&\ast&\bullet&\ast&\ast&\bullet\end{array}\right\}\begin{array}{rrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 5 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 7 & 8 & 8 & 8 & 8 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 19 & 19 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 5 & 7 & 8 & 6 & 2 & 0 & \rightarrow .\end{array}$$

130) [1,3,5,7,9,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \bullet & \ast & & \\ \ast & \ast & \bullet & \ast & & \\ \ast & \ast & \bullet & \ast & \ast & \ast \\ \bullet & \bullet & \bullet & \ast & \bullet & \bullet \\ \ast & \ast & \bullet & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 5 & 7 & 9 & 9 & 9 & 9 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 17 & 17 & 17 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 5 & 7 & 9 & 6 & 2 & 0 & \rightarrow & . \end{array}$$

175) [1,3,6,6,6,4,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \bullet & \ast & & & \\ \ast & \ast & \bullet & \ast & \ast & \ast \\ \ast & \ast & \bullet & \ast & \bullet & \bullet \\ \ast & \ast & \bullet & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 6 & 6 & 6 & 6 & 6 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 21 & 21 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 6 & 6 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

176) [1,3,6,6,6,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \bullet & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \ast & \bullet & \ast \\ \bullet & \ast & \ast & \bullet & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 6 & 6 & 6 & 6 & 6 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 21 & 21 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 6 & 6 & 5 & 2 & 0 & \rightarrow & . \end{array}$$

177) [1,3,6,6,6,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \bullet & \bullet & \ast & & & \\ \ast & \bullet & \ast & & & \\ \ast & \bullet & \ast & \ast & \ast & \bullet \\ \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 6 & 6 & 6 & 6 & 6 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 21 & 21 & 21 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 6 & 6 & 6 & 2 & 0 & \rightarrow & . \end{array}$$

196) [1,3,6,7,6,4,2]

$$\mathbb{Z}=\left\{\begin{array}{ccccc} \ast & \ast & \ast & & \\ \ast & \ast & \ast & & \\ \ast & \bullet & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \bullet \\ \ast & \ast & \ast & \ast & \bullet \\ \ast & \ast & \ast & \bullet & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 17 & 19 & 19 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 7 & 6 & 4 & 2 & 0 & \rightarrow & . \end{array}$$

197) [1,3,6,7,6,5,2]

$$\mathbb{Z}=\left\{\begin{array}{ccccc} \ast & \ast & \bullet & & \\ \ast & \ast & \ast & & \\ \ast & \ast & \bullet & \bullet & \ast \\ \bullet & \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \ast \end{array}\right\} \quad \begin{array}{rrrrrrrrrr} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 19 & 19 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 7 & 6 & 5 & 2 & 0 & \to . \end{array}$$

201) [1,3,6,7,7,4,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \bullet & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & \ast & \bullet & \bullet \\ \ast & \ast & \ast & \bullet & \ast & \ast \\ \ast & \ast & \ast & \bullet & \bullet & \bullet \end{array}\right\} \quad \begin{array}{rrrrrrrrrr} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 18 & 20 & 20 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 7 & 7 & 4 & 2 & 0 & \to . \end{array}$$

202) [1,3,6,7,7,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \bullet & \ast & \ast & \ast & \ast & \bullet \\ \ast & \bullet & \ast & \bullet & \bullet & \ast \\ \ast & \ast & \bullet & \bullet & \ast & \ast \end{array}\right\} \quad \begin{array}{rrrrrrrrrr} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 20 & 20 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 7 & 7 & 5 & 2 & 0 & \to . \end{array}$$

203) [1,3,6,7,7,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \bullet & & & \\ \ast & \ast & \bullet & & & \\ \ast & \ast & \ast & & & \\ \ast & \ast & \bullet & \ast & \ast & \bullet \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \ast & \bullet & \bullet & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{rrrrrrrrrr} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 7 & 7 & 7 & 7 & 7 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 20 & 20 & 20 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 7 & 7 & 6 & 2 & 0 & \to . \end{array}$$

207) [1,3,6,7,8,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \bullet & \bullet & \bullet & & & \\ \ast & \ast & \ast & & & \\ \ast & \bullet & \bullet & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \ast & \bullet & \ast & \ast & \ast & \ast \end{array}\right\} \quad \begin{array}{rrrrrrrrrr} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 7 & 8 & 8 & 8 & 8 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 18 & 19 & 19 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 7 & 8 & 5 & 2 & 0 & \to . \end{array}$$

208) [1,3,6,7,8,6,2]

$$\mathbb{Z}=\left\{\begin{matrix} * & * & \bullet & & & \\ * & * & * & & & \\ * & * & * & & & \\ \bullet & \bullet & * & * & * & * \\ \bullet & \bullet & * & \bullet & \bullet & \bullet \\ * & * & * & * & * & * \end{matrix}\right\} \begin{matrix} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 7 & 8 & 8 & 8 & 8 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 19 & 19 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 7 & 8 & 6 & 2 & 0 & \to . \end{matrix}$$

227) [1,3,6,8,6,4,2]

$$\mathbb{Z}=\left\{\begin{matrix} * & * & & \\ * & * & & \\ * & * & * & \bullet \\ * & * & * & * \\ * & * & \bullet & \bullet \\ * & * & \bullet & \bullet \\ * & \bullet & \bullet & \bullet \end{matrix}\right\} \begin{matrix} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 14 & 18 & 22 & 24 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 12 & 14 & 14 & 14 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 6 & 4 & 2 & 0 & \to . \end{matrix}$$

232) [1,3,6,8,7,4,2]

$$\mathbb{Z}=\left\{\begin{matrix} * & * & * & & & \\ * & * & * & & & \\ \bullet & * & * & & & \\ * & * & * & \bullet & * & \bullet \\ * & * & * & * & \bullet & \bullet \\ * & * & * & \bullet & \bullet & \bullet \end{matrix}\right\} \begin{matrix} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 17 & 19 & 19 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 7 & 4 & 2 & 0 & \to . \end{matrix}$$

233) [1,3,6,8,7,5,2]

$$\mathbb{Z}=\left\{\begin{matrix} * & \bullet & \bullet & & & \\ \bullet & * & \bullet & & & \\ \bullet & \bullet & * & & & \\ * & * & * & * & * & * \\ \bullet & \bullet & * & * & * & * \\ * & * & * & * & * & * \end{matrix}\right\} \begin{matrix} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \to \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \to \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 19 & 19 & \to \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 7 & 5 & 2 & 0 & \to . \end{matrix}$$

234) [1,3,6,8,7,6,2]

$$\mathbb{Z}=\left\{\begin{array}{ccccc} \ast & \bullet & \ast & & \\ \ast & \ast & \ast & & \\ \bullet & \bullet & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \ast \\ \ast & \ast & \ast & \bullet & \bullet \\ \ast & \ast & \ast & \bullet & \ast \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 18 & 18 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 7 & 6 & 2 & 0 & \rightarrow & . \end{array}$$

238) [1,3,6,8,8,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \bullet & \ast & & & \\ \ast & \ast & \ast & \bullet & \ast & \bullet \\ \ast & \bullet & \ast & \ast & \ast & \bullet \\ \ast & \bullet & \ast & \bullet & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 18 & 19 & 19 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 8 & 5 & 2 & 0 & \rightarrow & . \end{array}$$

239) [1,3,6,8,8,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \bullet & \ast & & & \\ \ast & \ast & \bullet & & & \\ \ast & \ast & \bullet & & & \\ \ast & \ast & \bullet & \ast & \ast & \bullet \\ \ast & \ast & \ast & \ast & \ast & \bullet \\ \bullet & \ast & \ast & \ast & \ast & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 8 & 8 & 8 & 8 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 19 & 19 & 19 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 8 & 6 & 2 & 0 & \rightarrow & . \end{array}$$

243) [1,3,6,8,9,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \bullet & \ast & & & \\ \ast & \bullet & \ast & & & \\ \ast & \bullet & \ast & & & \\ \ast & \bullet & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \bullet & \ast & \bullet \\ \ast & \bullet & \ast & \ast & \bullet & \bullet \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 9 & 9 & 9 & 9 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 17 & 18 & 18 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 9 & 5 & 2 & 0 & \rightarrow & . \end{array}$$

244) [1,3,6,8,9,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \ast & \ast & \ast & & & \\ \ast & \ast & \ast & & & \\ \ast & \bullet & \bullet & & & \\ \ast & \ast & \bullet & \ast & \ast & \ast \\ \bullet & \bullet & \bullet & \bullet & \ast & \bullet \\ \ast & \ast & \ast & \ast & \bullet & \ast \end{array}\right\} \quad \begin{array}{lcccccccccc} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow & \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 9 & 9 & 9 & 9 & \rightarrow & \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 18 & 18 & 18 & \rightarrow & \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 9 & 6 & 2 & 0 & \rightarrow & . \end{array}$$

249) [1,3,6,8,10,6,2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \bullet & \bullet & \bullet & & & \\ \bullet & * & * & & & \\ \bullet & * & * & & & \\ \bullet & * & \bullet & * & * & * \\ \bullet & * & \bullet & * & * & * \\ * & * & \bullet & * & * & * \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 8 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 17 & 17 & 17 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 8 & 10 & 6 & 2 & 0 & \rightarrow . \end{array}$$

268) [1, 3, 6, 9, 7, 4, 2]

$$\mathbb{Z} = \left\{\begin{array}{cccc} * & & & \\ * & & & \\ * & & & \\ * & * & * & * \\ * & * & * & \bullet \\ * & * & \bullet & \bullet \\ * & * & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 14 & 18 & 21 & 23 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 12 & 13 & 14 & 14 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 7 & 4 & 2 & 0 & \rightarrow . \end{array}$$

269) [1, 3, 6, 9, 7, 5, 2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & & \\ * & \bullet & \bullet & \bullet & & \\ * & * & \bullet & \bullet & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 13 & 16 & 17 & 17 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 7 & 5 & 2 & 0 & \rightarrow . \end{array}$$

274) [1, 3, 6, 9, 8, 5, 2]

$$\mathbb{Z} = \left\{\begin{array}{cccccc} * & * & * & & & \\ * & * & \bullet & & & \\ * & * & * & & & \\ * & * & * & \bullet & \bullet & \bullet \\ * & * & \bullet & \bullet & * & \bullet \\ * & * & * & * & \bullet & \bullet \end{array}\right\} \quad \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 17 & 18 & 18 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 8 & 5 & 2 & 0 & \rightarrow . \end{array}$$

275) [1,3,6,9,8,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\ast&\ast&\ast&&&\\ \bullet&\bullet&\bullet&&&\\ \bullet&\bullet&\bullet&&&\\ \ast&\ast&\ast&\ast&\ast&\ast\\ \ast&\ast&\ast&\ast&\bullet&\ast\\ \ast&\ast&\ast&\bullet&\ast&\bullet\end{array}\right\}\quad\begin{array}{lrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 18 & 18 & 18 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 8 & 6 & 2 & 0 & \rightarrow .\end{array}$$

279) [1,3,6,9,9,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\ast&\ast&\ast&&&\\ \ast&\bullet&\ast&&&\\ \ast&\bullet&\ast&&&\\ \ast&\ast&\ast&\ast&\bullet&\bullet\\ \ast&\bullet&\ast&\bullet&\bullet&\bullet\\ \ast&\ast&\ast&\ast&\bullet&\ast\end{array}\right\}\quad\begin{array}{lrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 17 & 18 & 18 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 9 & 5 & 2 & 0 & \rightarrow .\end{array}$$

280) [1,3,6,9,9,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\bullet\!\ast&\ast&\ast&&&\\ \ast&\ast&\ast&&&\\ \ast&\ast&\bullet&&&\\ \ast&\ast&\bullet&\ast&\ast&\bullet\\ \ast&\ast&\ast&\ast&\ast&\bullet\\ \bullet&\ast&\bullet&\bullet&\bullet&\ast\end{array}\right\}\quad\begin{array}{lrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 9 & 9 & 9 & 9 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 18 & 18 & 18 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 9 & 6 & 2 & 0 & \rightarrow .\end{array}$$

285) [1,3,6,9,10,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\bullet&\ast&\bullet&&&\\ \bullet&\ast&\bullet&&&\\ \ast&\ast&\ast&&&\\ \bullet&\ast&\bullet&\bullet&\ast&\bullet\\ \bullet&\ast&\ast&\ast&\ast&\ast\\ \bullet&\ast&\ast&\ast&\ast&\ast\end{array}\right\}\quad\begin{array}{lrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 10 & 10 & 10 & 10 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 17 & 17 & 17 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 10 & 6 & 2 & 0 & \rightarrow .\end{array}$$

290) [1,3,6,9,11,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc}\ast&\ast&\ast&&&\\ \bullet&\ast&\ast&&&\\ \bullet&\bullet&\ast&&&\\ \ast&\bullet&\bullet&\bullet&\bullet&\bullet\\ \ast&\ast&\bullet&\ast&\ast&\ast\\ \ast&\ast&\ast&\bullet&\bullet&\ast\end{array}\right\}\quad\begin{array}{lrrrrrrrrl}\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow\\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 11 & 11 & 11 & 11 & \rightarrow\\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 16 & 16 & 16 & \rightarrow\\ \mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 11 & 6 & 2 & 0 & \rightarrow .\end{array}$$

295) [1,3,6,9,12,6,2]

$$
\mathbb{Z} = \left\{ \begin{array}{cccccc}
\bullet & \bullet & \ast & & & \\
\bullet & \ast & \ast & & & \\
\bullet & \bullet & \bullet & & & \\
\ast & \bullet & \ast & \bullet & \ast & \ast \\
\ast & \bullet & \ast & \ast & \bullet & \ast \\
\bullet & \ast & \ast & \ast & \ast & \bullet
\end{array} \right\}
\quad
\begin{array}{lrrrrrrrrl}
\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\
\mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 9 & 12 & 12 & 12 & 12 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 15 & 15 & 15 & \rightarrow \\
\mathbf{H}_{A}: & 1 & 3 & 6 & 9 & 12 & 6 & 2 & 0 & \rightarrow .
\end{array}
$$

314) [1,3,6,10,7,4,2]

$$
\mathbb{Z} = \left\{ \begin{array}{cccc}
\ast & & & \\
\ast & & & \\
\ast & & & \\
\bullet & \bullet & \bullet & \bullet \\
\ast & \bullet & \bullet & \bullet \\
\ast & \ast & \bullet & \bullet \\
\ast & \ast & \ast & \bullet \\
\ast & \ast & \ast & \ast
\end{array} \right\}
\quad
\begin{array}{lrrrrrrrrl}
\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 14 & 18 & 21 & 23 & \rightarrow \\
\mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 11 & 12 & 13 & 13 & \rightarrow \\
\mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 7 & 4 & 2 & 0 & \rightarrow .
\end{array}
$$

320) [1,3,6,10,8,5,2]

$$
\mathbb{Z} = \left\{ \begin{array}{cccc}
\ast & & & \\
\bullet & & & \\
\ast & & & \\
\ast & \ast & \bullet & \ast \\
\ast & \ast & \ast & \bullet \\
\bullet & \ast & \bullet & \bullet \\
\ast & \ast & \bullet & \bullet \\
\ast & \bullet & \bullet & \ast
\end{array} \right\}
\quad
\begin{array}{lrrrrrrrrl}
\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 14 & 18 & 21 & 23 & \rightarrow \\
\mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 12 & 13 & 13 & 13 & \rightarrow \\
\mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 8 & 5 & 2 & 0 & \rightarrow .
\end{array}
$$

321) [1,3,6,10,8,6,2]

$$
\mathbb{Z} = \left\{ \begin{array}{ccccc}
\ast & \ast & \ast & & \\
\bullet & \ast & \ast & & \\
\ast & \bullet & \ast & \bullet & \bullet \\
\ast & \ast & \ast & \bullet & \bullet \\
\ast & \ast & \bullet & \bullet & \bullet \\
\ast & \ast & \ast & \ast & \bullet
\end{array} \right\}
\quad
\begin{array}{lrrrrrrrrl}
\mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 26 & \rightarrow \\
\mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\
\mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 13 & 16 & 16 & 16 & \rightarrow \\
\mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 8 & 6 & 2 & 0 & \rightarrow .
\end{array}
$$

325) [1,3,6,10,9,5,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} * & * & * & & & \\ \bullet & * & * & & & \\ \bullet & * & * & & & \\ * & * & * & \bullet & \bullet & \bullet \\ \bullet & * & * & \bullet & * & * \\ \bullet & * & * & \bullet & \bullet & * \end{array}\right\} \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 16 & 17 & 17 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 9 & 5 & 2 & 0 & \rightarrow . \end{array}$$

326) [1,3,6,10,9,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} * & \bullet & * & & & \\ * & * & * & & & \\ * & * & * & & & \\ * & \bullet & \bullet & \bullet & \bullet & * \\ * & * & * & \bullet & * & \bullet \\ \bullet & * & * & * & \bullet & \bullet \end{array}\right\} \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 14 & 17 & 17 & 17 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 9 & 6 & 2 & 0 & \rightarrow . \end{array}$$

331) [1,3,6,10,10,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} * & * & \bullet & & & \\ \bullet & * & * & & & \\ * & \bullet & \bullet & & & \\ * & \bullet & * & * & * & * \\ * & * & \bullet & * & * & * \\ * & \bullet & \bullet & \bullet & * & \bullet \end{array}\right\} \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 17 & 17 & 17 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 10 & 6 & 2 & 0 & \rightarrow . \end{array}$$

326) [1,3,6,10,11,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \bullet & \bullet & \bullet & & & \\ * & * & * & & & \\ * & * & \bullet & & & \\ * & * & * & * & * & * \\ \bullet & * & \bullet & * & * & * \\ * & * & \bullet & \bullet & * & \bullet \end{array}\right\} \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 11 & 11 & 11 & 11 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 16 & 16 & 16 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 11 & 6 & 2 & 0 & \rightarrow . \end{array}$$

341) [1,3,6,10,12,6,2]

$$\mathbb{Z}=\left\{\begin{array}{cccccc} \bullet & \bullet & * & & & \\ \bullet & * & * & & & \\ \bullet & * & * & & & \\ * & \bullet & * & \bullet & \bullet & * \\ * & * & * & \bullet & \bullet & \bullet \\ \bullet & * & \bullet & * & * & * \end{array}\right\} \begin{array}{lrrrrrrrrl} \mathbf{H}_{\mathbb{Z}}: & 1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & \rightarrow \\ \mathbf{H}_{\mathbb{X}}: & 1 & 3 & 6 & 10 & 12 & 12 & 12 & 12 & \rightarrow \\ \mathbf{H}_{\mathbb{Y}}: & 1 & 3 & 6 & 10 & 15 & 15 & 15 & 15 & \rightarrow \\ \mathbf{H}_{A}: & 1 & 3 & 6 & 10 & 12 & 6 & 2 & 0 & \rightarrow . \end{array}$$

Appendix A

The total 341 Artinian cases based on

Theorem of Fröberg and Laksov in [3].

1]	1, 3, 2, 2, 2, 2, 2	2]	1, 3, 3, 2, 2, 2, 2	3]	1, 3, 3, 3, 2, 2, 2
4]	1, 3, 3, 3, 3, 2, 2	5]	1, 3, 3, 3, 3, 3, 2	6]	1, 3, 3, 4, 2, 2, 2
7]	1, 3, 3, 4, 3, 2, 2	8]	1, 3, 3, 4, 3, 3, 2	9]	1, 3, 3, 4, 4, 2, 2
10]	1, 3, 3, 4, 4, 3, 2	11]	1, 3, 3, 4, 4, 4, 2	12]	1, 3, 3, 4, 5, 2, 2
13]	1, 3, 3, 4, 5, 3, 2	14]	1, 3, 3, 4, 5, 4, 2	15]	1, 3, 3, 4, 5, 5, 2
16]	1, 3, 3, 4, 5, 6, 2	17]	1, 3, 4, 2, 2, 2, 2	18]	1, 3, 4, 3, 2, 2, 2
19]	1, 3, 4, 3, 3, 2, 2	20]	1, 3, 4, 3, 3, 3, 2	21]	1, 3, 4, 4, 2, 2, 2
22]	1, 3, 4, 4, 3, 2, 2	23]	1, 3, 4, 4, 3, 3, 2	24]	1, 3, 4, 4, 4, 2, 2
25]	1, 3, 4, 4, 4, 3, 2	26]	1, 3, 4, 4, 4, 4, 2	27]	1, 3, 4, 4, 5, 2, 2
28]	1, 3, 4, 4, 5, 3, 2	29]	1, 3, 4, 4, 5, 4, 2	30]	1, 3, 4, 4, 5, 5, 2
31]	1, 3, 4, 4, 5, 6, 2	32]	1, 3, 4, 5, 2, 2, 2	33]	1, 3, 4, 5, 3, 2, 2
34]	1, 3, 4, 5, 3, 3, 2	35]	1, 3, 4, 5, 4, 2, 2	36]	1, 3, 4, 5, 4, 3, 2
37]	1, 3, 4, 5, 4, 4, 2	38]	1, 3, 4, 5, 5, 2, 2	39]	1, 3, 4, 5, 5, 3, 2
40]	1, 3, 4, 5, 5, 4, 2	41]	1, 3, 4, 5, 5, 5, 2	42]	1, 3, 4, 5, 5, 6, 2
43]	1, 3, 4, 5, 6, 2, 2	44]	1, 3, 4, 5, 6, 3, 2	45]	1, 3, 4, 5, 6, 4, 2
46]	1, 3, 4, 5, 6, 5, 2	47]	1, 3, 4, 5, 6, 6, 2	48]	1, 3, 5, 2, 2, 2, 2
49]	1, 3, 5, 3, 2, 2, 2	50]	1, 3, 5, 3, 3, 2, 2	51]	1, 3, 5, 3, 3, 3, 2
52]	1, 3, 5, 4, 2, 2, 2	53]	1, 3, 5, 4, 3, 2, 2	54]	1, 3, 5, 4, 3, 3, 2
55]	1, 3, 5, 4, 4, 2, 2	56]	1, 3, 5, 4, 4, 3, 2	57]	1, 3, 5, 4, 4, 4, 2
58]	1, 3, 5, 4, 5, 2, 2	59]	1, 3, 5, 4, 5, 3, 2	60]	1, 3, 5, 4, 5, 4, 2
61]	1, 3, 5, 4, 5, 5, 2	62]	1, 3, 5, 4, 5, 6, 2	63]	1, 3, 5, 5, 2, 2, 2
64]	1, 3, 5, 5, 3, 2, 2	65]	1, 3, 5, 5, 3, 3, 2	66]	1, 3, 5, 5, 4, 2, 2
67]	1, 3, 5, 5, 4, 3, 2	68]	1, 3, 5, 5, 4, 4, 2	69]	1, 3, 5, 5, 5, 2, 2
70]	1, 3, 5, 5, 5, 3, 2	71]	1, 3, 5, 5, 5, 4, 2	72]	1, 3, 5, 5, 5, 5, 2
73]	1, 3, 5, 5, 5, 6, 2	74]	1, 3, 5, 5, 6, 2, 2	75]	1, 3, 5, 5, 6, 3, 2
76]	1, 3, 5, 5, 6, 4, 2	77]	1, 3, 5, 5, 6, 5, 2	78]	1, 3, 5, 5, 6, 6, 2
79]	1, 3, 5, 6, 2, 2, 2	80]	1, 3, 5, 6, 3, 2, 2	81]	1, 3, 5, 6, 3, 3, 2
82]	1, 3, 5, 6, 4, 2, 2	83]	1, 3, 5, 6, 4, 3, 2	84]	1, 3, 5, 6, 4, 4, 2
85]	1, 3, 5, 6, 5, 2, 2	86]	1, 3, 5, 6, 5, 3, 2	87]	1, 3, 5, 6, 5, 4, 2
88]	1, 3, 5, 6, 5, 5, 2	89]	1, 3, 5, 6, 5, 6, 2	90]	1, 3, 5, 6, 6, 2, 2
91]	1, 3, 5, 6, 6, 3, 2	92]	1, 3, 5, 6, 6, 4, 2	93]	1, 3, 5, 6, 6, 5, 2

94]	1, 3, 5, 6, 6, 6, 2	95]	1, 3, 5, 6, 7, 2, 2	96]	1, 3, 5, 6, 7, 3, 2
97]	1, 3, 5, 6, 7, 4, 2	98]	1, 3, 5, 6, 7, 5, 2	99]	1, 3, 5, 6, 7, 6, 2
100]	1, 3, 5, 7, 2, 2, 2	101]	1, 3, 5, 7, 3, 2, 2	102]	1, 3, 5, 7, 3, 3, 2
103]	1, 3, 5, 7, 4, 2, 2	104]	1, 3, 5, 7, 4, 3, 2	105]	1, 3, 5, 7, 4, 4, 2
106]	1, 3, 5, 7, 5, 2, 2	107]	1, 3, 5, 7, 5, 3, 2	108]	1, 3, 5, 7, 5, 4, 2
109]	1, 3, 5, 7, 5, 5, 2	110]	1, 3, 5, 7, 5, 6, 2	111]	1, 3, 5, 7, 6, 2, 2
112]	1, 3, 5, 7, 6, 3, 2	113]	1, 3, 5, 7, 6, 4, 2	114]	1, 3, 5, 7, 6, 5, 2
115]	1, 3, 5, 7, 6, 6, 2	116]	1, 3, 5, 7, 7, 2, 2	117]	1, 3, 5, 7, 7, 3, 2
118]	1, 3, 5, 7, 7, 4, 2	119]	1, 3, 5, 7, 7, 5, 2	120]	1, 3, 5, 7, 7, 6, 2
121]	1, 3, 5, 7, 8, 2, 2	122]	1, 3, 5, 7, 8, 3, 2	123]	1, 3, 5, 7, 8, 4, 2
124]	1, 3, 5, 7, 8, 5, 2	125]	1, 3, 5, 7, 8, 6, 2	126]	1, 3, 5, 7, 9, 2, 2
127]	1, 3, 5, 7, 9, 3, 2	128]	1, 3, 5, 7, 9, 4, 2	129]	1, 3, 5, 7, 9, 5, 2
130]	1, 3, 5, 7, 9, 6, 2	131]	1, 3, 6, 2, 2, 2, 2	132]	1, 3, 6, 3, 2, 2, 2
133]	1, 3, 6, 3, 3, 2, 2	134]	1, 3, 6, 3, 3, 3, 2	135]	1, 3, 6, 4, 2, 2, 2
136]	1, 3, 6, 4, 3, 2, 2	137]	1, 3, 6, 4, 3, 3, 2	138]	1, 3, 6, 4, 4, 2, 2
139]	1, 3, 6, 4, 4, 3, 2	140]	1, 3, 6, 4, 4, 4, 2	141]	1, 3, 6, 4, 5, 2, 2
142]	1, 3, 6, 4, 5, 3, 2	143]	1, 3, 6, 4, 5, 4, 2	144]	1, 3, 6, 4, 5, 5, 2
145]	1, 3, 6, 4, 5, 6, 2	146]	1, 3, 6, 5, 2, 2, 2	147]	1, 3, 6, 5, 3, 2, 2
148]	1, 3, 6, 5, 3, 3, 2	149]	1, 3, 6, 5, 4, 2, 2	150]	1, 3, 6, 5, 4, 3, 2
151]	1, 3, 6, 5, 4, 4, 2	152]	1, 3, 6, 5, 5, 2, 2	153]	1, 3, 6, 5, 5, 3, 2
154]	1, 3, 6, 5, 5, 4, 2	155]	1, 3, 6, 5, 5, 5, 2	156]	1, 3, 6, 5, 5, 6, 2
157]	1, 3, 6, 5, 6, 2, 2	158]	1, 3, 6, 5, 6, 3, 2	159]	1, 3, 6, 5, 6, 4, 2
160]	1, 3, 6, 5, 6, 5, 2	161]	1, 3, 6, 5, 6, 6, 2	162]	1, 3, 6, 6, 2, 2, 2
163]	1, 3, 6, 6, 3, 2, 2	164]	1, 3, 6, 6, 3, 3, 2	165]	1, 3, 6, 6, 4, 2, 2
166]	1, 3, 6, 6, 4, 3, 2	167]	1, 3, 6, 6, 4, 4, 2	168]	1, 3, 6, 6, 5, 2, 2
169]	1, 3, 6, 6, 5, 3, 2	170]	1, 3, 6, 6, 5, 4, 2	171]	1, 3, 6, 6, 5, 5, 2
172]	1, 3, 6, 6, 5, 6, 2	173]	1, 3, 6, 6, 6, 2, 2	174]	1, 3, 6, 6, 6, 3, 2
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TABLE 13

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Abstract

Artinian level algebra of codimension 3 and type 2

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Based on a theorem of Fröberg and Laksov, I researched about 341 Artinian O-sequences which have a possibility that becomes a level of Codimension 3, Type 2, and Length 7.

After the O-sequence, is not become a level by Theorem and Remark, was found out, the table, which conforms to each case, was completed. Of the O-sequence that was not rejected by the former method, the special case, that is not become a level, was also demonstrated.

the 58 sequences that become a level organized "link-sum" with CoCoA and S-plus.